Error Analysis and Accuracy Assessment principle in Gravity Field Approximation

PAGravf4.5; https://www.zcyphygeodesy.com/en/
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March 2026, Beijing, 100830, China

原理

The Earth's gravity field is a continuous physical field permeating the entire Earth-fixed space. Consequently, gravity field elements can only be mathematically expressed as spatial averages over grids of finite resolution. This intrinsic characteristic fundamentally distinguishes gravity field elements from discrete geometric geodetic elements, such as point coordinates in a geodetic control network. As a result, error analysis and quality control for gravity field approximation cannot directly adopt the intuitive and point-to-point effective methods used in geometric geodesy.

However, all gravity field elements and their interrelationships exhibit high analytical correlation throughout the Earth's external space. The gravity field approximation problem is, in essence, a one-dimensional linear transformation between gravity field elements. By fully leveraging these intrinsic properties, it is possible to construct multiple analytical function constraints, significantly enriching the conceptual framework and strategic toolkit for error analysis and quality control in gravity field modeling.

1. Concepts of Observational Error and Target Element Accuracy

Gravity field elements are continuously differentiable in the Earth's external space or on boundary surfaces. Thus, although observation field elements are acquired at discrete points, the "observation field elements" functioning within gravity field approximation models are effectively the spatial averages of these observations at a specific resolution.

Spatial Domain: For instance, spatial-domain integration requires grid-averaged field elements on the boundary surface.

Spectral Domain: Truncating observational models at a maximum degree is mathematically equivalent to treating observation field elements as spatial averages at a corresponding resolution.

Similarly, target gravity field elements derived via approximation represent spatial averages corresponding to their effective resolution, not discrete point values.

Example: A 1′×1′ grid of approximated gravity disturbances represents the mean disturbance over each grid cell.

Principle: When computing a target element via spectral basis function expansion (e.g., spherical harmonics), series truncation implies the result is equivalent to the spatial average at the effective resolution.

Therefore, in the context of gravity field approximation, the definitions are as follows:

(a) Observational element Error:

Specifically refers to the error of its spatial average at a given resolution. This is termed the Spatial Representativeness Error, distinct from the point-wise observation error.

(b) Target Element Accuracy:

Refers to the accuracy of its spatial average at the effective resolution. This is termed the Spatial Representativeness Accuracy, rather than point-wise accuracy.

The spatial representativeness error of observational field elements depends on observation error, point distribution density, and local field structure. When observation accuracy is sufficiently high, this error is dominated by point density and local structure.

Example: If point observed gravity accuracy is 0.1 mGal but the representativeness error for a 1′×1′ gravity disturbance is 1 mGal, the latter is governed by point distribution density, field structure, and gridding algorithm performance. In such scenarios, further improving point observed gravity accuracy does not reduce the representativeness error. Historically, given the correlation between short-wavelength gravity field structures and undulating terrain, this was often termed Terrain Representativeness Error.

The spatial representativeness accuracy of a gravity field element grid depends on the approximation fidelity, spatial resolution, and the local gravity field structure (particularly short- and ultra-short-wavelength components) within each cell.

Important Reminder: The spatial resolution of gravity field elements is not a trivial geometric pixel average; its representativeness is intrinsically coupled to the local gravity field structure.

(a) Identical resolutions offer stronger representativeness in simple fields and weaker representativeness in complex ones.

(b) Field elements between grid cells contain rich spatial analytical functional relationships. Consequently, the information content within such a grid far exceeds what its geometric resolution alone suggests.

2. Theoretical Basis for Error Analysis and Accuracy Assessment

Earth's gravity field approximation is essentially a one-dimensional linear transformation within the gravity field's linear space. Field elements of the same type at different locations, different types, and various elements at different locations are all analytically correlated. These correlations can be analytically expressed via spatial-domain integral formulas or spectral-domain parameterized basis function expansions. Thus, Earth's gravity field signals possess high analyticity, pervasive throughout the external space. Leveraging this distinctive geophysical property allows for the extraction of weak signals from environments with strong observational noise – a feature distinguishing gravity field approximation from general geometric geodesy. For instance, based on this property, satellite gravity methods can effectively extract gravity field signals from environments with signal-to-noise ratios below 10^{-3}.

(1) When employing spatial-domain integral methods, two primary schemes exist for assessing the accuracy of local gravity field modeling:

Scheme I: Direct Derivation Based on the Integral Mathematical Model

This approach relies directly on the gravity field integral formula:

(a) Estimate Spatial Representativeness Error: Estimate the spatial representativeness error of the gridded observed field elements on the integration boundary (e.g., the gravity anomaly grid in Stokes' formula), accounting for errors introduced during discretization and gridding.

(b) Derive Error Propagation Formula: Derive an error propagation formula from the gravity field integral formula.

(c) Estimate Target Accuracy: Estimate the spatial representativeness accuracy of the target element, neglecting reference model errors.

Scheme II: Indirect Assessment Based on Analytical Functional Relationships

This approach is based on the analytical functional relationships between gravity field elements:

(a) Construct an Integral Algorithm: Formulate an algorithm where the target element is the integrand, its surface is the integration boundary, and discrete observed field elements serve as input reference truth values.

(b) Compute and Compare: Compute integral values at observation points. Statistically analyze the discrepancy between these integral values (possessing spatial representativeness accuracy) and the reference truth values (discrete observation accuracy). This yields the spatial representativeness error of the observed field elements via discrepancy statistics.

(c) Estimate Target Accuracy: Utilize the derived representativeness error from Step (b), following the procedure of Scheme I, to estimate the spatial representativeness accuracy of the target field elements.

(2) When approximating the gravity field using the spectral-domain least squares method with heterogeneous observations:

Such as in constructing Global Geopotential Models (GGMs) or Spherical Radial Basis Function (SRBF) models, error analysis and accuracy assessment can be rigorously implemented within the least squares framework. The specific procedural steps are:

(a) Residual Analysis and Extraction of Representativeness Error: Using the estimated spectral coefficients derived from the approximation, compute the estimated values for each type of observations at the observation points via spectral series expansion. Statistically analyze the residuals between these estimated values (which possess spatial representativeness accuracy) and the corresponding observations (treated as reference truth with their measurement accuracy). This yields the spatial representativeness error for each type of observations.

(b) Derivation of Error Propagation Formulas: Deduce the error propagation algorithm directly from the mathematical model of the spectral-domain least squares approximation.

(c) Computation of Accuracies: Utilizing the spatial representativeness errors of the various observations obtained in the previous step, apply the error propagation algorithm to compute the accuracy metrics of all spectral coefficients and the spatial representativeness accuracy of the target field elements.

Evidently, gravity field approximation methods can effectively determine the spatial representativeness error for any type of observed gravity field element, regardless of its origin (spaceborne, airborne, terrestrial, or marine). However, these methods cannot directly quantify the intrinsic accuracy of the measurement instrument. Gravity field approximation provides only an upper bound for the actual instrument error. Yet, it is the Spatial Representativeness Error that plays the decisive role in governing the fidelity and resolution of the final gravity field model.

3. Fundamental Principles for Regional Gravity Field and Geoid Modeling

The precise determination of geoidal height at any point mandates global coverage of observational gravity field data. Regional datasets alone are insufficient to fully capture the medium- to long-wavelength components of the gravity field and the corresponding geoid undulations. In gravity field approximation, the accuracy of the recovered gravity field and geoid generally improves with increasing wavelength (i.e., larger spatial scales).

(1) The hierarchical control for Regional Gravity Field and Geoid Modeling

Based on this premise, a scientifically robust modeling principle of regional gravity field and geoid in PAGravf4.5: Employ an appropriate Global Geopotential Model (GGM) as both the reference field and the far-zone boundary condition. This step is critical for mitigating medium- to long-wavelength errors. Upon this foundation, all available regional and surrounding observational resources are integrated to enhance the accuracy of the medium- to short-wavelength components. The synergy of global reference control and local data integration enables high-precision modeling of the regional gravity field and geoid.

By analogous logic, for localized domains such as provinces or cities, a regional gravity field model should serve as the reference field to control the accumulation of medium- to short-wavelength errors. Local and surrounding data are then integrated, with the external regional model acting as the far-zone boundary constraint. This strategy aims to refine the short- to ultra-short-wavelength components of the geoid, achieving fine-scale modeling for the local area.

Consequently, in compliance with geodetic control principles, regional gravimetric geoid refinement schemes should adhere to a hierarchical control principle: "**The refinement of a gravimetric geoid for a sub-region shall utilize a gravity field model of a larger encompassing region as both the reference field and the far-zone boundary condition.**"

(2) GNSS/Leveling Data Fusion Requirements

The spirit leveling, which transfers height differences station-by-station, essentially measures the geopotential difference between leveling benchmarks. In principle, the relative error of this difference is independent of the total leveling route length.

For any specific region, there exists a critical spatial scale threshold. Below this threshold, the relative accuracy of gravimetric height anomaly differences between two points may be inferior to that of GNSS/leveling height anomaly differences. Therefore, algorithms fusing GNSS/leveling data with the gravimetric geoid should effectively integrate the high-precision medium- to long-wavelength components from gravity field modeling with the high-precision short- to ultra-short-wavelength components from the GNSS/leveling control network. Regarding error management, such algorithms must simultaneously suppress short-wavelength noise in gravimetric height anomalies while controlling long-distance error accumulation in GNSS/leveling height anomalies.

Furthermore, GNSS/leveling height anomalies, once properly referenced to an Earth-fixed system and reduced to the global height datum, qualify as general observational field elements. As such, they can be rigorously incorporated into the gravity field approximation and geoid modeling process alongside other heterogeneous data sources.