Height Systems and Height Datum: Theory and Concepts

PAGravf4.5; https://www.zcyphygeodesy.com/en/
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The fundamental essence of geodetic elevation is the geopotential difference. In geodesy, elevation serves as the geometric approximation of the geopotential number in Earth's gravity field space, defined in an Earth-fixed coordinate reference system. By selecting the geoid potential W_G as the constant reference datum, the geopotential number of a terrain point represents its potential energy relative to the geoid. This constitutes a physically meaningful "physical elevation", an objective quantity inherent to the gravity field. The corresponding height system is termed the Geopotential Number System.

7.11.1 Rigorous Geodetic Definitions of Height Systems

The definition of geometric geodetic elevation must satisfy three fundamental conditions:

● Uniqueness: The elevation of any point must be single-valued.

● Measurability: Reduction corrections should be minimal to avoid significant deviations from observed leveling elevations, in local, low-order levelings.

● Equipotential Consistency: Points on an equipotential surface should possess almost identical elevations.

Geodetic elevation (orthometric or normal height) is the geometric realization of the geopotential number within the Earth-fixed reference system, adhering to the constraints of uniqueness and measurability. The height difference between two points represents the geometric realization of their geopotential difference. The two primary forms are Orthometric Height and Normal Height, both referred to collectively as "geometric elevations."

Let W_A be the geopotential of terrain point A. Let point Q share the same ellipsoidal coordinates (latitude, longitude) as A, with its normal potential U_Q equal to W_A. By gravity field theory, the segment QA represents the height anomaly ζ_A at point A (see Figure 7.9).

All elements in Figure 7.9 reside within a unified Earth-fixed reference system with a consistent geometric scale. Arrows indicate the direction of measurement. This establishes the Earth-fixed reference system theory as the indispensable foundation for modern height systems.

(1) Physical-Geodetic Definition of the Orthometric Height System

Orthometric height h^* is defined as the ratio of the geopotential number c_A of point A to the mean gravity $\bar{g}_A$ between A and the geoid G:

$$h_A^* = \frac{W_G - W_A}{\bar{g}_A} = \frac{c_A}{\bar{g}_A} \tag{11.1}$$

The mean gravity $\bar{g}_A$ is physically defined as:

$$\bar{g}_A = \frac{1}{h_A^*}\int_0^{h^*} g(h)dh \tag{11.2}$$

where dh is the line element between the terrain surface and the geoid. Assuming a constant crustal density ρ, the mean gravity $\bar{g}_A$ can be approximated using the Prey reduction formula:

$$\bar{g}_A = g_A - \left(\frac{1}{2}\frac{\partial\gamma}{\partial h} + 2\pi G\rho\right) h_A^* \tag{11.3}$$

where g_A is the observed gravity at A, G is the gravitational constant, and $\partial\gamma/\partial h$ is the normal gravity gradient. The resulting elevation is known as the Helmert Orthometric Height.

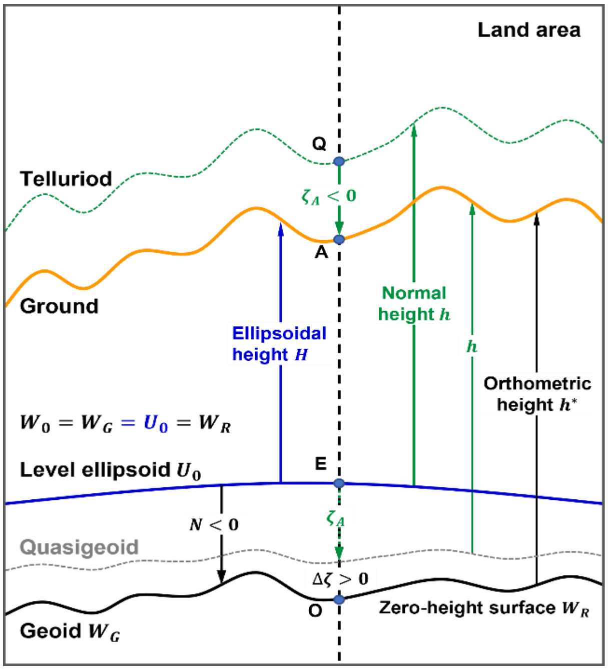

Figure 7.9: Geometric Relationships among Ellipsoidal Height, Orthometric/Normal Height, and Geoidal Height in an Earth-Fixed Reference System

(2) Physical-Geodetic Definition of the Normal Height System

Normal height h is defined as the ratio of the normal geopotential number of point Q ($U_0 - U_Q$) to the mean normal gravity $\bar{\gamma}_Q$ between Q and the normal ellipsoid E:

$$h_A = \frac{U_0 - U_Q}{\bar{\gamma}_Q} \tag{11.4}$$

According to the Molodensky condition, the normal geopotential number at Q equals the actual geopotential number at A ($U_0 - U_A = c_A = W_G - W_A$). Substituting this yields the Molodensky Normal Height:

$$h_A = \frac{U_0 - U_Q}{\bar{\gamma}_Q} = \frac{W_G - W_A}{\bar{\gamma}_Q} = \frac{C_A}{\bar{\gamma}_Q} \tag{11.5}$$

where $\bar{\gamma}_Q$ is the Molodensky mean normal gravity. Given $W_G = U_0$, it follows that

$$U_Q = W_A \tag{11.6}$$

Equation (11.6) confirms that point Q lies on the equipotential surface passing through A (Fig. 7.9). China's national height system is based on this definition.

Note: The Molodensky condition here is distinct from the Molodensky Boundary Value Problem; it is the geometric realization of Bruns' formula at point A.

Since $\bar{\gamma}_Q$ ($\bar{g}_A$) is constant, normal (orthometric) height is also unique, path-independent, and measurable.

7.11.2 Concept of the Analytical Geoid and Analytical Orthometric Height

The so-called "true" geoid (or terrain-corrected geoid), which attempts to account for the topographic mass effects through reduction, suffers from decimeter-level uncertainties in continental mountainous regions. These uncertainties stem from necessary approximations in terrain density models and assumptions regarding methods of terrain mass reduction (e.g., Helmert condensation or complete Bouguer reduction). At the centimeter-level precision demanded, such a geoid fails to meet the fundamental constraints of uniqueness and precise measurability required for valid geodetic elements. Consequently, the geoidal height derived following such terrain adjustments is metrologically undefined; it is neither uniquely determinable nor does it possess an accepted concept of accuracy.

To resolve this fundamental theoretical and practical impasse, PAGravf4.5 introduces the Analytical Geoid, a well-defined geodetic element achieved via specific, theoretically consistent terrain mass reduction that preserves the potential field's invariance. Crucially, before and after this mass reduction, the geopotential (or the disturbing potential T) remains invariant everywhere across the entire ground surface and throughout the external space (or any closed surface outside the geoid). This geoidal height is defined as the Analytical Geoidal Height, which is mathematically equivalent to the analytical continuation of the height anomaly (ζ) from the ground surface (or exterior) down to the geoid.

In metrological sense, helmert orthometric height fails to qualify as a general geodetic element satisfying the uniqueness and measurability requirement. Furthermore, it does not meet the datum conditions necessary for GNSS Replacing Leveling, since rendering the relationship ($H = h^* + N$) between ellipsoidal height (H), orthometric height (h^*), and geoidal height (N) ambiguous.

PAGravf4.5 addresses this long-standing deficiency by introducing the Analytical Orthometric Height. In this formulation, the gravity at any point along the theoretical path from the ground to the geoid is not approximated by a constant or a simplified density model. Instead, it is defined as the gravity value obtained via the analytical continuation of the external gravity field to that specific point (i.e., the analytical gravity). The mean gravity along this path is then rigorously defined as the geometric mean of these continuously varying analytical gravity values. The orthometric height derived from this rigorously defined mean gravity constitutes the Analytical Orthometric Height.

Unlike the Helmert orthometric height, both the Analytical Orthometric Height and the Normal Height strictly satisfy the datum conditions for GNSS Replacing Leveling. And the fundamental equation $H = h$ (normal height) + ζ holds with analytical rigor. Furthermore, these two height systems share a rigorous analytical functional relationship within the framework of gravity field theory. This robust and consistent theoretical foundation is not Earth-specific, two height systems can be directly extended and applied to other celestial bodies, such as the Moon and terrestrial planets.

The Analytical Orthometric Height requires no assumptions regarding crustal density. Its mean gravity can be continuously refined and updated using the latest gravity field data and models, making it a dynamic and improvable geodetic element. From the perspective of the uniqueness and measurability of geodetic elements, the Analytical Orthometric Height is more suitable for modern height datum purposes than other types of orthometric heights.

Simple calculations using global geopotential models demonstrate that, numerically, the values of the Analytical Orthometric Height and the Normal Height are globally much closer to each other than either is to the Helmert orthometric height. The discrepancy between the Analytical Orthometric Height and the Helmert orthometric height can reach approximately 60 cm at an elevation of 3,000 meters.

7.11.3 Analytical Functional Relationship between Orthometric and Normal Heights

Geodetic elevation (orthometric or normal height) is the geometric expression of geopotential in the Earth-fixed reference system. Both orthometric and normal heights are objectively unique and precisely measurable.

The ellipsoidal height H relates to orthometric height h^* and geoidal height (geoid undulation) N, as well as normal height h and height anomaly ζ, via:

$$H = h^* + N = h + \zeta \tag{11.7}$$

Equation (11.7) provides the theoretical basis for determining h^* (or h) using GNSS-derived H and a geoid model (N or ζ). The difference between orthometric and normal heights is:

$$h^* - h = \zeta - N = \Delta\zeta \tag{11.8}$$

That is, the discrepancy between the orthometricl height h^* and normal height h equals the difference between the height anomaly ζ and geoidal height N.

The integral solution to the Stokes Boundary Value Problem determines the disturbing potential everywhere outside the geoid, yielding both ζ and N (Generalized Stokes Formula). Crucially, this solution constrains the analytical relationship between ζ and N (Zhang Chuanyin, 2017):

$$\zeta = N + \Delta\zeta = N + \int_0^{h^*} \frac{\partial \zeta}{\partial h} dh = N - \int_0^{h^*} \frac{\delta g}{\gamma} dh \tag{11.9}$$

where δg is the gravity disturbance and γ are is the normal gravity along the line element dh.

7.11.4 Applicability of the Geoid as the Zero-Elevation Surface

While orthometric and normal heights are unique and measurable geometric quantities, their physical representation (the geopotential number) involves approximations. This is an inherent characteristic of their geodetic definitions.

(1) Geodetic Basis for the Uniqueness of the Height Datum

If the normal height of point A is zero ($h_A = 0$), Substituting equation (11.5), we can obtain:

$$h_A = \frac{U_0 - U_Q}{\bar{\gamma}_Q} = \frac{W_G - W_A}{\bar{\gamma}_Q} = \frac{c_A}{\bar{\gamma}_Q} = 0 \quad \Longrightarrow \quad c_A = 0, \quad W_A = W_G \tag{11.10}$$

This indicates that a point with zero normal height has zero geopotential number, thus the point should lie on the geoid ($W_A = W_G$).

Substituting $c_A = 0$ into the orthometric height definition (11.1) yields:

$$h_A^* = \frac{W_G - W_A}{\bar{g}_A} = \frac{c_A}{\bar{g}_A} = \frac{0}{\bar{g}_A} = 0 \tag{11.11}$$

Equation (11.11) shows that the orthometric height of point A also equals zero, $h_A^* = 0$.

Consequently, a point with zero normal height also has zero orthometric height.

Conclusion: The zero-orthometric-height surface, zero-normal-height surface, zero-geopotential-number surface, and the Geoid are coincident. They all share the constant geopotential W_G. Therefore, regardless of the specific height system (Orthometric Height, Normal Height, or Geopotential Number), the geodetic height starting datum surface is only the (global or regional) Geoid.

(2) Analysis of Geoidal Geopotential Properties and Datum Nature

The geodetic definitions of orthometric and normal heights [Eqs. (11.1) and (11.5)] constitute the sole theoretical foundation in geodesy for elucidating the nature of the height starting datum and geodetic elevation. All formulas for leveling elevation difference corrections are derived from these definitions.

The definition of height system can be uniformly expressed as:

$$h = \frac{W_G - W}{g} = \frac{c}{g} \tag{11.12}$$

where:

○ If g represents the mean gravity between the terrain point and the geoid, Eq. (11.12) defines the Orthometric Height System.

○ If g represents the Molodensky mean normal gravity, it defines the Molodensky Normal Height System.

○ If $g = 1$, it defines the Geopotential Number System.

A unique and time-invariant reference datum is a mandatory constraint for the entire geodetic discipline. Examining Eq. (11.12), since $g \neq 0$, the geopotential W of a terrain point is the sole independent variable determining its orthometric (or normal) height, whereas the geoidal geopotential W_G is a pre-defined constant. Consequently, Eq. (11.12) dictates that only the constant W_G can serve as the unique, invariant starting datum value for orthometric and normal height systems.

(3) Definition of Geoidal Geometric Deformation

The elevation of any terrain point is an objective quantity uniquely defined by its geopotential number. For a deforming Earth, the geopotential W changes objectively with time due to the redistribution of internal mass, inducing temporal variations in the geopotential number (and thus geometric elevation) at any point. Earth's deformation directly causes discrepancies in the spatial distribution of geopotential between two time epochs. This means that in the Earth-fixed reference system, the geometric positions of the geoid (W_G remains unchanged) at two epochs differ, and this difference represents the geoid's geometric deformation.

7.11.5 Issues with the Quasi-Geoid as a Datum Surface

In traditional physical geodesy, the closed surface where the ellipsoidal height equals the ground height anomaly is termed the quasi-geoid. Historically, this surface was regarded as the datum for normal heights. However, this perception contradicts the rigorous definition of normal height [Eq. (11.5)] for two reasons:

(a) Physical Inconsistency: The geopotential number on the zero-normal-height surface is zero; thus, the zero-normal-height surface is the geoid, not the quasi-geoid.

(b) Violation of Uniqueness: For two points with identical latitude and longitude but different attitudes, their height anomalies (ζ) differ. If normal heights were referenced to the quasi-geoid, there would exist two non-coincident starting points in the vertical direction for the same location, violating the uniqueness requirement of the normal height system.

Actual observation points rarely lie precisely on the specific Digital Elevation Model (DEM) surface employed to construct the ground height anomaly model. For centimeter-level applications, a height-dependent correction $\delta\zeta$, accounting for the height anomaly gradient (or gravity disturbance), should be applied:

$$\zeta = \zeta_0 + \delta\zeta = \zeta_0 + \int_{h_0}^{h} \frac{\partial\zeta}{\partial h} dh = \zeta_0 - \int_{h_0}^{h} \frac{\delta g}{\gamma} dh = \zeta_0 - \left[\frac{\delta g}{\gamma}\right](h - h_0) \tag{11.13}$$

where:

ζ: Height anomaly at the actual observation point (elevation h).

ζ_0: Height anomaly interpolated from the ground height anomaly model to the point's horizontal location.

h_0: Terrain elevation interpolated from the DTM.

$\delta g, \gamma$: Gravity disturbance and normal gravity, respectively, along the vertical segment.

[·]: Averaging operator.

Practical Implication: For instance, using a 1′×1′ ground height anomaly model to represent the quasi-geoid can result in correction terms $\delta\zeta$ reaching decimeter-level magnitudes in regions like western China. Thus, while the normal height system is theoretically rigorous, looking the quasi-geoid as its datum surface fails to meet geodetic requirements at decimeter-level accuracy and contradicts Eq. (11.5).

Consequently, PAGravf4.5 comprehensively deemphasizes the concept of the quasi-geoid and related legacy knowledge.

7.11.6 Geometric Properties of Height Systems and Conceptual Updates

Both orthometric and normal height systems are strictly defined within the Earth-fixed reference system based on gravity field theory.

(1) Geometric Parallelism of Orthometric Heights:

Consider two surfaces of constant orthometric height, h_1' and h_2', such that their difference $\Delta h_{12}' = h_2' - h_1' = C \neq 0$ is constant. From Eq. (11.7), the ellipsoidal height difference ΔH_{12} between the two surfaces is:

$$\Delta H_{12} = H_2 - H_1 = (h_2' + N) - (h_1' + N) = h_2' - h_1' = \Delta h_{12}' = C \tag{11.14}$$

Eq. (11.14) demonstrates that the ellipsoidal height difference between two constant orthometric height surfaces equals their orthometric height difference, independent of the geoidal height N. This implies that orthometric equiheight surfaces are parallel within the Earth-fixed reference system.

● **Revised Definition:** Orthometric height is rigorously the distance from a terrain point to the geoid measured along a straight line perpendicular to the geoid. A direct corollary is that the integration path in the mean gravity definition [Eq. (11.2)] should be treated as a straight line.

● **Correction of Misconception:** The conventional view that orthometric height is the irregular curvilinear distance along the plumb line is incorrect. Due to the deflection of the vertical and the curvature of the normal gravity line, the plumb line is an irregular curve whose length exceeds the straight-line distance. This traditional interpretation lacks a rigorous geodetic foundation.

● **Limitation:** Globally parallel closed surfaces do not exist in external Earth space. Parallelism holds only in infinitesimally local spaces; thus, orthometric height possesses typical local properties.

(2) Geometric Behavior of Normal Heights:

Normal height is also unique and measurable. However, since the height anomaly ζ varies with attitude (elevation), normal equiheight surfaces are not strictly parallel in the Earth-fixed reference system. The signal of the height anomaly attenuates with increasing elevation. Consequently, the geometric shape of a normal equiheight surface becomes progressively smoother relative to the geoid as elevation increases.

(3) Conclusion:

● Orthometric Height: Offers intuitive geometric measurement properties due to the parallelism of its equipheight surfaces with the geoid.

● Normal Height: aligns more closely with gravity field properties, as its equipheight surfaces smooth out with elevation increase, reflecting the attenuation of gravitational signals.

● Both height systems possess distinct advantages, limitations, and scientific applicability. Their coexistence is both necessary and scientifically justified.