Integral Algorithm Formulas for the Anomalous Earth Gravity Field

PAGravf4.5; https://www.zcyphygeodesy.com/en/

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While the global gravity field is typically modeled in the spectral domain, spatial integral algorithms are essential for local gravity field approximation. To perform integration over a finite radius in local regions, the Remove-Restore Method, based on a reference global geopotential model, is standard practice:

(1) Remove: Compute and subtract the model values of the anomalous field elements on the boundary surface to obtain residual field elements.

(2) Integrate: Apply a spatial integral algorithm with a finite radius to these residuals to derive the target residual field element at the computation point.

(3) Restore: Add the model value of the target element at the computation point to obtain the final local approximation.

7.9.1 Generalized Stokes and Hotine Integral Formulas

Given the gravity anomaly Δg on the geoid or an external equipotential surface S the disturbing potential T or height anomaly ζ at an external point $T(\theta,\lambda,r)$ is computed via the Generalized Stokes Integral:

$$T(\theta,\lambda,r)=\gamma\zeta(\theta,\lambda,r)=\frac{1}{4\pi}\iint_{s}\Delta g' S(r,\psi,r')ds \tag{9.1}$$

where r' is the geocentric distance of the moving element ds, and $S(r,\psi,r')$ is the Generalized Stokes Kernel:

$$S(r,\psi,r')=\frac{2}{L}+\frac{1}{r}-\frac{3L}{r^2}-\frac{5r'cos\psi}{r^2}-\frac{3r'}{r^2}cos\psi ln\frac{r-r'cos\psi+L}{2r} \tag{9.2}$$

where L being the Euclidean distance between the computation and moving points.

Singularity Handling: When the computation point coincides with the moving point, the integral becomes singular. The analytical value for this singular term is:

$$\zeta|_0=\frac{A_0}{\gamma}\Delta g_0 \tag{9.3}$$

where A_0, Δg_0, and γ are the area, gravity anomaly, and normal gravity at the computation point, respectively.

Similarly, given the gravity disturbance δg on S , the Generalized Hotine Integral is used:

$$T(\theta,\lambda,r)=\gamma\zeta(\theta,\lambda,r)=\frac{1}{4\pi}\iint_{s}\delta g' H(r,\psi,r')ds \tag{9.4}$$

where the Generalized Hotine Kernel $H(r,\psi,r')$ is:

$$H(r,\psi,r')=\frac{2}{L}-\frac{1}{r'}ln\frac{r-r'cos\psi+L}{r(1-cos\psi)} \tag{9.5}$$

The corresponding singular value is:

$$\zeta|_0 = \frac{A_0}{\gamma}\delta g_0 \tag{9.6}$$

FFT Implementation: By approximating the geocentric distances r and r' as constant mean values, these integrals transform into convolution forms, enabling rapid computation via the Fast Fourier Transform (FFT) algorithm.

Note: The Stokes boundary value problem strictly requires the boundary surface S to be an equipotential surface (e.g., the geoid which can be constructed from a global geopotential model up to degree ~360). For equipotential surfaces within an altitude of 10 km, a normal (or orthometric) equiheight surface may serve as a valid approximation.

7.9.2 Generalized Vening-Meinesz Integral Formulas

Taking horizontal derivatives of the generalized Stokes formula (9.1) in a local horizontal coordinate system yields the Generalized Vening-Meinesz Formulas for vertical deflections (ξ, η):

$$\xi = -\frac{1}{4\pi r\gamma}\iint_S \Delta g' \frac{\partial S(r,\psi,r')}{\partial\psi}\frac{\partial\psi}{\partial\varphi}ds,\quad \eta = -\frac{1}{4\pi r cos\varphi\gamma}\iint_S \Delta g' \frac{\partial S(r,\psi,r')}{\partial\psi}\frac{\partial\psi}{\partial\lambda}ds \tag{9.7}$$

From the relation:

$$cos\psi = sin\varphi sin\varphi' + cos\varphi cos\varphi' cos(\lambda' - \lambda) \tag{9.8}$$

taking horizontal derivatives on both sides gives:

$$-sin\psi\frac{\partial\psi}{\partial\varphi} = cos\varphi sin\varphi' - sin\varphi cos\varphi' cos(\lambda' - \lambda) \tag{9.9}$$

$$-sin\psi\frac{\partial\psi}{\partial\lambda} = cos\varphi cos\varphi' sin(\lambda' - \lambda) \tag{9.10}$$

Using spherical trigonometric formulas:

$$sin\psi cos\alpha = cos\varphi sin\varphi' - sin\varphi cos\varphi' cos(\lambda' - \lambda) \tag{9.11}$$

$$sin\psi sin\alpha = cos\varphi' sin(\lambda' - \lambda) \tag{9.12}$$

Combining Equations (9.9) through (9.12), we have:

$$\frac{\partial\psi}{\partial\varphi} = -cos\alpha,\quad \frac{\partial\psi}{\partial\lambda} = -cos\varphi sin\alpha \tag{9.13}$$

Substituting into Equation (9.7) yields:

$$\xi = \frac{1}{4\pi r\gamma}\iint_S \Delta g' \frac{\partial S(r,\psi,r')}{\partial\psi} cos\alpha ds,\quad \eta = \frac{1}{4\pi r\gamma}\iint_S \Delta g' \frac{\partial S(r,\psi,r')}{\partial\psi} sin\alpha ds \tag{9.14}$$

Considering $= \sqrt{r^2 + r'^2 - 2rr'cos\psi}$, we have:

$$\frac{\partial}{\partial\psi}L = \frac{rr'}{L}sin\psi,\quad \frac{\partial}{\partial\psi}\left(\frac{1}{L}\right) = -\frac{1}{L^2}\frac{\partial}{\partial\psi}L = -\frac{rr'}{L^3}sin\psi \tag{9.15}$$

$$\frac{\partial}{\partial\psi}ln\frac{r-r'cos\psi+L}{2r} = \frac{1}{r-r'cos\psi+L}\left(\frac{rr'}{L}sin\psi + r'sin\psi\right) = \frac{r'sin\psi}{r+L-r'cos\psi}\frac{L+r}{L} \tag{9.16}$$

$$\begin{aligned}\frac{\partial}{\partial\psi}S(r,\psi,r') &= \frac{\partial}{\partial\psi}\left(\frac{2}{L} + \frac{1}{r} - \frac{3L}{r^2} - \frac{5r'cos\psi}{r^2} - \frac{3r'cos\psi}{r^2}ln\frac{r-r'cos\psi+L}{2r}\right)\\ &= \frac{\partial}{\partial\psi}\frac{2}{L} - \frac{3}{r^2}\frac{\partial}{\partial\psi}L + \frac{5r'sin\psi}{r^2} + \frac{3r'sin\psi}{r^2}ln\frac{r+L-r'cos\psi}{2r} - \frac{3r'cos\psi}{r^2}\frac{\partial}{\partial\psi}ln\frac{r+L-r'cos\psi}{2r}\\ &= \left(-\frac{2rr'}{L^3} - \frac{3r'}{rL} + \frac{5r'}{r^2} + \frac{3r'}{r^2}ln\frac{r-r'cos\psi+L}{2r} - \frac{3r'cos\psi}{r^2}\frac{r'}{r-r'cos\psi+L}\frac{L+r}{L}\right)sin\psi\\ &= \left[-\frac{2r}{L^3} - \frac{3}{rL} + \frac{5}{r^2} + \frac{3}{r^2}ln\frac{r-r'cos\psi+L}{2r} - \frac{3r'(L+r)cos\psi}{r^2L(r-r'cos\psi+L)}\right]r'sin\psi\end{aligned} \tag{9.17}$$

Similarly, deriving from the Hotine formula (9.4) using gravity disturbances δg:

$$\xi = \frac{1}{4\pi r\gamma}\iint_S \delta g' \frac{\partial H(r,\psi,r')}{\partial \psi} cos\alpha ds, \quad \eta = \frac{1}{4\pi r\gamma}\iint_S \delta g' \frac{\partial H(r,\psi,r')}{\partial \psi} sin\alpha ds \tag{9.18}$$

Since: $\frac{\partial}{\partial\psi} ln\frac{r-r'cos\psi+L}{r(1-cos\psi)} = \frac{r(1-cos\psi)}{r-r'cos\psi+L}\frac{\left(\frac{rr'}{L}sin\psi+r'sin\psi\right)r(1-cos\psi)+(r-r'cos\psi+L)rsin\psi}{r^2(1-cos\psi)^2}$

$$= \frac{sin\psi}{r-r'cos\psi+L}\frac{\frac{L+r}{L}r'(1-cos\psi)+(r-r'cos\psi+L)}{1-cos\psi} = \left[\frac{r'(L+r)}{(r-r'cos\psi+L)L} + \frac{1}{1-cos\psi}\right] sin\psi \tag{9.19}$$

Therefore: $\frac{\partial}{\partial\psi} H(r,\psi,r') = \frac{\partial}{\partial\psi}\left(\frac{2}{L} - \frac{1}{r'} ln\frac{r-r'cos\psi+L}{r(1-cos\psi)}\right) = \frac{\partial}{\partial\psi}\frac{2}{L} - \frac{1}{r'}\frac{\partial}{\partial\psi} ln\frac{r-r'cos\psi+L}{r(1-cos\psi)}$

$$= \left[-\frac{2rr'}{L^3} - \frac{L-r}{(r-r'cos\psi+L)L} + \frac{1}{r'(1-cos\psi)}\right] sin\psi \tag{9.20}$$

These formulas allow the computation of vertical deflections at any point on the Earth's surface or in external space from gravity anomalies or disturbances defined on an equipotential surface. Like the Stokes/Hotine integrals, they can be accelerated using FFT under the constant-radius approximation.

7.9.3 Poisson Integral Algorithm and Applications

The Poisson Integral solves the First Boundary Value Problem (Dirichlet Problem), performing analytical continuation of an anomalous field element μ (e.g., δg or ζ) from a source surface S to a target surface D at a different attitude:

$$\mu(\theta,\lambda,r) = \frac{1}{4\pi r}\int_S (\theta',\lambda',r')\frac{r^2-r'^2}{L^3} ds \tag{9.21}$$

Singularity neutralization:

When $r \to r'$ (computation point on the source surface), the kernel becomes undefined (0/0) and singular ($L \to 0$). To resolve this, we apply an identity transformation (Hofmann, 2006) to obtain the Modified Poisson Integral:

$$\mu(\theta,\lambda,r) = \frac{r'^2}{r^2}\mu(\theta,\lambda,r') + \frac{1}{4\pi r}\int_S [\mu(\theta',\lambda',r') - \mu(\theta',\lambda',r)]\frac{r^2-r'^2}{L^3} ds \tag{9.22}$$

Here, the difference term $[\mu - \mu'] \to 0$ as $L \to 0$, effectively neutralizing the singularity of the kernel and ensuring numerical stability.

Application to Gravity Gradients:

The radial disturbing gravity gradient T_{rr} can be derived by taking the radial derivative of the Poisson integral for gravity disturbance δg.

Applying the Poisson integral to the gravity disturbance δg yields:

$$\delta g(\theta,\lambda,r) = \frac{1}{4\pi r}\iint_S \delta g' \frac{r^2-r'^2}{L^3} ds \tag{9.23}$$

Considering $T_{rr} = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}T\right) = -\frac{\partial}{\partial r}(\delta g)$, taking the radial partial derivative of both sides of Equation (9.23) gives:

$$T_{rr} = -\frac{1}{4\pi r}\iint_S \delta g' \frac{\partial}{\partial r}\frac{r^2-r'^2}{L^3} ds$$

$$= \frac{1}{4\pi r}\iint_S \delta g' \frac{r^3-5rr'^2+(r^2+3r'^2)r'^2cos\psi}{L^5} ds \tag{9.24}$$

Similar modification techniques (subtracting the computation point value inside the integral) should be applied to Eq. (9.24) to suppress singularities when computing gradients on or near the source surface.

7.9.4 Forward and Inverse Integral Operations for the Anomalous Gravity Field

(1) Computing Gravity Disturbance from Height Anomaly

Differentiating the Poisson integral for the disturbing potential T along the vertical direction yields the gravity disturbance δg :

$$\delta g = \frac{\partial T}{\partial n} \approx -\frac{\gamma \partial \zeta}{\partial r} = -\frac{\gamma}{2\pi} \iint_s \frac{\zeta - \zeta_p}{l^3} ds \tag{9.25}$$

where ∂n denotes differentiation along the vertical direction, and l is the straight-line distance between the computation point and the moving element on the equipotential boundary.

Singularity: When the computation point coincides with the moving point, the singular value is:

$$\delta g|_0 = \frac{\gamma \sqrt{A_0/\pi}}{4} \left(\zeta_{xx} + \zeta_{yy}\right) \tag{9.26}$$

where ζ_{xx}, ζ_{yy} are the second-order horizontal derivatives of the height anomaly.

Eq. (9.25) is known as the Inverse Hotine Integral. It computes gravity disturbance on an equipotential surface from height anomalies defined on the same surface.

Note: The boundary must be an equipotential surface.

(2) Computing Gravity Anomaly from Height Anomaly

Substituting the fundamental equation of physical geodesy into Eq. (9.25) yields the Inverse Stokes Integral:

$$\varDelta g = -\frac{\gamma}{2\pi} \iint_s \frac{\zeta - \zeta_p}{l^3} ds - \frac{\zeta \gamma}{2r} \tag{9.27}$$

This formula derives gravity anomaly from height anomaly on an equipotential surface.

(3) Computing Height Anomaly from Vertical Deflection

$$\zeta = \frac{r}{4\pi} \iint_\sigma ctg\frac{\psi}{2} (\xi cos\alpha + \eta sin\alpha)\, d\sigma \tag{9.28}$$

Singularity:

$$\zeta|_0 = \frac{A_0}{4\pi} \left(\xi_y + \eta_x\right) \tag{9.29}$$

where ξ_y, η_x are are the cross-derivatives of the vertical deflection components.

(4) Computing Gravity Anomaly from Vertical Deflection

$$\varDelta g = -\frac{\gamma}{4\pi} \iint_\sigma \left(3csc\psi - csc\psi csc\frac{\psi}{2} - tg\frac{\psi}{2}\right) (\xi cos\alpha + \eta sin\alpha)\, d\sigma \tag{9.30}$$

Singularity:

$$\varDelta g|_0 = -\frac{\gamma \sqrt{A_0/\pi}}{4} \left(\xi_y + \eta_x\right) \tag{9.31}$$

(5) Computing Gravity Disturbance from Vertical deflection Integral

Combining the fundamental differential equation with Eqs. (9.28) and (9.30) yields:

$$\delta g = -\frac{\gamma}{4\pi}$$

$$\iint_\sigma \left(3csc\psi - csc\psi csc\frac{\psi}{2} - tg\frac{\psi}{2} - 2ctg\frac{\psi}{2}\right) (\xi cos\alpha + \eta sin\alpha)\, d\sigma \tag{9.32}$$

Singularity:

$$\delta g|_0 = -\frac{\gamma}{2\pi} \left(\sqrt{\pi A_0} + \frac{A_0}{r}\right) \left(\xi_y + \eta_x\right) \tag{9.33}$$

Eqs. (9.28), (9.30), and (9.32) constitute the Inverse Vening-Meinesz Integrals. Under the constant-radius approximation ($r \approx$const), all inverse formulas (9.25 – 9.32) can be efficiently computed using the FFT algorithm.

(6) Forward and Inverse Computation of Disturbing Gravity Gradients

Forward (Gradient to Disturbance): Compute δg at an external point from T_{rr} on an equipotential surface using the Generalized Hotine Integral:

$$\delta g(\theta,\lambda,r) = \frac{1}{4\pi}\iint_{s} T_{rr}H(r,\psi,r')ds \tag{9.34}$$

Inverse (Disturbance to Gradient): Compute T_{rr} on the surface from δg via the radial gradient integral:

$$T_{rr} = \frac{1}{2\pi}\iint \frac{\delta g-\delta g'}{l^3}ds \tag{9.35}$$

🌏 **Optimization Strategy:**

Optimal combination of these algorithms with "1-to-2-step residual cumulative approximation" scheme can enhance the accuracy and stability of boundary value problem solutions and short-wavelength field approximations.

7.9.5 Analytical Properties of Gravity Field Integral Kernel Functions

The kernel functions depend on the straight-line distance l, expressible via the spherical angular distance ψ. Two major challenges arise in numerical integration:

- Spectral Leakage: Due to convergence issues in the kernel functions.
- Singularity: When the computation point lies on the boundary surface ($\psi \to 0$), leading to resolution-dependent jumps in numerical results.

(1) Analysis under Spherical Approximation ($r' = r = R$):

Let $l = 2R sin(\psi/2)$. The kernel functions on the boundary sphere are:

- Stokes Kernel:

$$\begin{aligned} S(\psi) = R\cdot S(R,\psi,R) &= sin^{-1}\frac{\psi}{2} + 1 - 6sin\frac{\psi}{2} - 5cos\psi - 3cos\psi ln\frac{1-cos\psi+2sin(\psi/2)}{2} \\ &= 1 + sin^{-1}\frac{\psi}{2} - 6sin\frac{\psi}{2} - 5cos\psi - 3cos\psi ln\left(sin\frac{\psi}{2} + sin^2\frac{\psi}{2}\right) \end{aligned} \tag{9.36}$$

- Hotine kernel:

$$H(\psi) = R\cdot H(R,\psi,R) = 2 - ln\left(1 + sin^{-1}\frac{\psi}{2}\right) \tag{9.37}$$

- Vening-Meinesz Kernel (Derivative of Hotine):

$$V(\psi) = \frac{\partial}{\partial\psi}H(\psi) = \frac{1}{2}\frac{ctg\frac{\psi}{2}}{1+sin\frac{\psi}{2}} \tag{9.38}$$

As shown in Figure 7.7, all three kernels diverge to infinity as $\psi \to 0$ ($S,H,V \to \infty$), necessitating singularity handling.

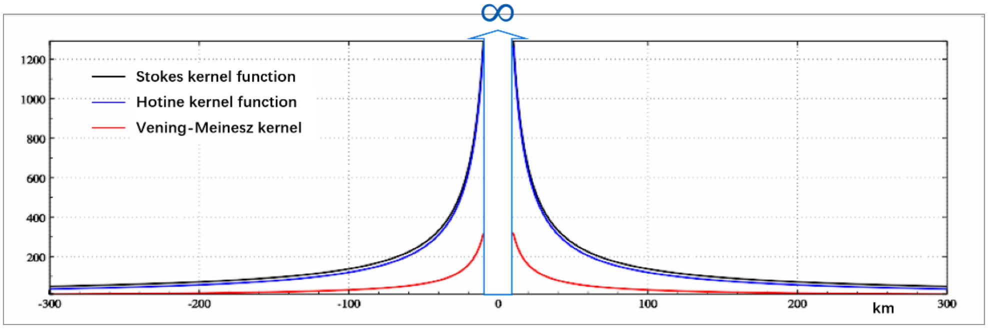

Figure 7.7: Curves of Major Gravity Field Integral Spherical Kernel Functions

(2) The Poisson Kernel Instability:

For the Poisson integral (Eq. 9.21) used for same-type continuation:

● If $r = r'$, the kernel $P(\psi) = (r^2 - r'^2)/L^3$ becomes identically zero/undefined.

● If points coincide ($L \to 0$), it becomes singular (0/0 form).

Consequently, direct Poisson integration on the same surface is numerically unstable. Effective analytical continuation requires restricting the integration range and employing modified forms (e.g., Eq. 9.22) to neutralize singularities.

(3) Recommendation on Kernel Modification:

While the Poisson integral theoretically underpins Stokes, Hotine, and Vening-Meinesz solutions, its severe high-order oscillations and non-convergence on the boundary introduce uncontrollable uncertainties. Historically, two approaches were used to mitigate this:

● Assuming isotropic random statistical properties for residual fields.

● Kernel Modification: Altering kernel functions based on spectral approximations.

However, these modification techniques rely on assumptions (statistical or data-driven) that lack a rigorous foundation in analytical gravity field theory, compromising their universality. Consequently, PAGravf4.5 does not recommend the use of gravity field integral kernel modification algorithms.

Despite numerical challenges, spatial integral formulas remain indispensable for explicitly expressing the analytical relationships between gravity field elements, serving as the cornerstone of physical geodesy and gravity field approximation theory.