# 地面大地测量负荷影响格林函数积分算法

## 1、区域负荷直接影响积分公式

①地面及地球外部重力位直接影响积分公式

已知地面等效水高$h_w$，则地面及地球外部重力位的直接影响$V_w$，可由万有引力公式直接给出：

$$V_w = G\rho_w \int_S \frac{h_w}{L} dS,\ L = \sqrt{r^2 + r'^2 - 2rr'cos\psi} \tag{1}$$

式中：$L$为地面或地球外部计算点$(r,\theta,\lambda)$与地面流动点$(r',\theta',\lambda')$间的空间距离；$dS$为地面流动积分面元；$\psi$为计算点$(r,\theta,\lambda)$与地面流动点$(r',\theta',\lambda')$间的球面角距。

$$cos\psi = cos\theta cos\theta' + sin\theta sin\theta' cos(\lambda' - \lambda),\ sin\psi = sin\theta cos\theta' + cos\theta sin\theta' cos(\lambda' - \lambda) \tag{2}$$

$$sin\psi cos\alpha = sin\theta cos\theta' - cos\theta sin\theta' cos(\lambda' - \lambda),\ sin\psi sin\alpha = sin\theta' sin(\lambda' - \lambda) \tag{3}$$

$$\frac{\partial\psi}{\partial\theta} = -\frac{\partial\psi}{\partial\varphi} = cos\alpha,\ \frac{\partial\psi}{\partial\lambda} = -sin\alpha sin\theta \tag{4}$$

在地面，当计算点与流动点重合时，有：

$$L = r\psi,\ r - r'cos\psi = r\psi^2/2 \tag{5}$$

$$A = dS = r^2 \int_{\alpha=0}^{2\pi} \int_0^{\psi_0} \psi d\psi d\alpha = \pi r^2 \psi_0^2 \rightarrow \psi_0 = \frac{1}{r}\sqrt{\frac{A}{\pi}} \tag{6}$$

式中：$A = dS$为流动积分面元的面积。

此时，积分（1）在计算点处奇异：

$$V_0 = G\rho_w r^2 \int_{\alpha=0}^{2\pi} \int_0^{\psi_0} \frac{h_w}{r\psi} \psi d\psi d\alpha = 2\pi G\rho_w h_w r\psi_0 \tag{7}$$

②地面及地球外部扰动重力直接影响积分公式

根据扰动重力定义式，有：

$$\delta g = -\frac{\partial V_w}{\partial r} = -G\rho_w \int_S h_w \frac{\partial}{\partial r}\left(\frac{1}{L}\right) dS = G\rho_w \int_S h_w \frac{r - r'cos\psi}{L^3} dS \tag{8}$$

在地面上，当计算点与流动点重合时，积分（8）奇异：

$$\delta g_0 = 2\pi G\rho_w h_w \int_0^{\psi_0} \frac{\psi^2/2}{\psi^3} \psi d\psi = \pi G\rho_w h_w \psi_0 \tag{9}$$

③地面及地球外部垂线偏差直接影响积分公式

$$\text{总垂线偏差}\Theta = \frac{1}{\gamma r}\frac{\partial V_w}{\partial\psi} = \frac{G\rho_w}{\gamma r} \int_S h_w \frac{\partial}{\partial\psi}\left(\frac{1}{L}\right) dS = -\frac{G\rho_w}{\gamma} \int_S h_w r' \frac{sin\psi}{L^3} dS \tag{10}$$

$$\xi = \Theta\frac{\partial\psi}{\partial\theta} = -\frac{G\rho_w}{\gamma} \int_S h_w r' \frac{sin\psi}{L^3} cos\alpha dS,\ \eta = -\Theta\frac{\partial\psi}{\partial\lambda} = -\frac{G\rho_w}{\gamma} sin\theta \int_S h_w r' \frac{sin\psi}{L^3} sin\alpha dS \tag{11}$$

④地面及地球外部扰动重力梯度直接影响积分公式

根据扰动重力梯度定义式，有：

$$T_{nn} = -T_{rr} = -\frac{\partial^2 V_w}{\partial r^2} = G\rho_w \int_S h_w \frac{\partial}{\partial r}\left(\frac{r - r'cos\psi}{L^3}\right) dS = G\rho_w \int_S h_w \left[\frac{1}{L^3} - \frac{3(r - r'cos\psi)^2}{L^5}\right] dS \tag{12}$$

$$\frac{\partial}{\partial r}\left(\frac{r - r'cos\psi}{L^3}\right) = \frac{1}{L^3} - \frac{3(r - r'cos\psi)}{L^4}\frac{\partial}{\partial r}L = \frac{1}{L^3} - \frac{3(r - r'cos\psi)^2}{L^5},\ \frac{\partial}{\partial r}L = \frac{r - r'cos\psi}{L}$$

在地面上，当计算点与流动点重合时，积分（12）奇异：

$$T_{nn}^{0}=2\pi G\rho_w h_w r^2\int_0^{\psi_0}\left(\frac{1}{r^3\psi^3}-\frac{3\psi^4}{4r^3\psi^5}\right)\psi d\psi=-\frac{\pi G\rho_w h_w}{r}\left(\frac{1}{\psi_0^2}+\frac{3}{2}\psi_0\right)\approx-\frac{\pi G\rho_w h_w}{r\psi_0^2} \quad (13)$$

⑤地面及地球外部水平重力梯度直接影响积分公式

$$\text{总水平梯度}\Gamma=\frac{\partial^2 V_w}{r^2\partial\psi^2}=-\frac{G\rho_w}{r}\int_S h_w r'\frac{\partial}{\partial\psi}\left(\frac{sin\psi}{L^3}\right)dS=-\frac{G\rho_w}{r}\int_S h_w r'\left(\frac{cos\psi}{L^3}-\frac{3rr'sin^2\psi}{L^5}\right)dS \quad (14)$$

$$T_{\varphi\varphi}=-\Gamma\frac{\partial^2\psi}{\partial\theta^2}=\frac{G\rho_w}{r}\int_S h_w r'\left(\frac{cos\psi}{L^3}-\frac{3rr'sin^2\psi}{L^5}\right)ctg\psi(1-cos\alpha)dS \quad (15)$$

$$T_{\lambda\lambda}=\Gamma\frac{\partial^2\psi}{\partial\lambda^2}=\frac{G\rho_w}{r}\int_S h_w r'\left(\frac{cos\psi}{L^3}-\frac{3rr'sin^2\psi}{L^5}\right)\left[ctg\psi-ctg\psi(sin\theta sin\alpha)^2-\frac{cos\theta cos\theta'}{sin\psi}\right]dS \quad (16)$$

$$\frac{\partial^2\psi}{\partial\theta^2}=\frac{\partial}{\partial\theta}cos\alpha=\frac{\partial}{\partial\theta}\frac{sin\theta cos\theta'-cos\theta sin\theta' cos(\lambda'-\lambda)}{sin\psi}=ctg\psi(1-cos^2\alpha) \quad (17)$$

$$\frac{\partial^2\psi}{\partial\lambda^2}=-sin\theta\frac{\partial}{\partial\lambda}sin\alpha=-sin\theta sin\theta'\frac{\partial}{\partial\lambda}\frac{sin(\lambda'-\lambda)}{sin\psi}=sin\theta sin\theta'\left[\frac{cos(\lambda'-\lambda)}{sin\psi}-\frac{sin(\lambda'-\lambda)cos\psi}{sin^2\psi}sin\alpha sin\theta\right]$$

$$=\frac{cos\psi-cos\theta cos\theta'}{sin\psi}-\frac{cos\psi}{sin\psi}(sin\theta sin\alpha)^2=(1-sin^2\theta sin^2\alpha)ctg\psi-\frac{cos\theta cos\theta'}{sin\psi} \quad (18)$$

**2、负荷间接影响格林函数及积分公式**

地表单位点质量负荷$q_w=\rho_w h_w$（kg/m$^2$）对大地测量参数的影响，可用负荷格林函数表示。令$t=cos\psi$，则地面大地测量参数的负荷间接影响格林函数算法公式分别为：

① 地面重力位间接影响格林函数$G_i^V$或高程异常间接影响格林函数$G_i^\zeta$：

$$G_i^V(\psi)=\gamma G_i^\zeta(\psi)=\frac{a\gamma}{M}\frac{k'_\infty}{2sin\frac{\psi}{2}}+\frac{a\gamma}{M}\sum_{n=0}^{\infty}(k'_n-k'_\infty)P_n(t) \quad (19)$$

②地面重力间接影响格林函数$G_i^g$：

$$G_i^g(\psi)=-\frac{\gamma}{M}\frac{k'_\infty-2h'_\infty}{2sin\frac{\psi}{2}}-\frac{\gamma}{M}\sum_{n=0}^{\infty}[(n+1)k'_n-k'_\infty-2(h'_n-h'_\infty)]P_n(t) \quad (20)$$

③扰动重力间接影响格林函数$G_i^{\delta g}$：

$$G_i^{\delta g}(\psi)=-\frac{\gamma}{M}\frac{k'_\infty}{2sin\frac{\psi}{2}}-\frac{\gamma}{M}\sum_{n=0}^{\infty}[(n+1)k'_n-k'_\infty]P_n(t) \quad (21)$$

④地倾斜间接影响格林函数$G_i^t$：

$$G_i^t(\psi)=-\frac{1}{M}\frac{h'_\infty cos\frac{\psi}{2}}{4sin^2\frac{\psi}{2}}+\frac{1}{M}\frac{k'_\infty cos\frac{\psi}{2}\left(1+2sin\frac{\psi}{2}\right)}{2sin\frac{\psi}{2}\left(1+sin\frac{\psi}{2}\right)}-\frac{1}{M}\sum_{n=1}^{\infty}\left(k'_n-\frac{k'_\infty}{n}-h'_n+h'_\infty\right)\frac{\partial P_n(t)}{\partial\psi} \quad (22)$$

⑤垂线偏差间接影响格林函数$G_i^\Theta$：

$$G_i^\Theta(\psi)=\frac{1}{M}\frac{k'_\infty cos\frac{\psi}{2}\left(1+2sin\frac{\psi}{2}\right)}{2sin\frac{\psi}{2}\left(1+sin\frac{\psi}{2}\right)}-\frac{1}{M}\sum_{n=1}^{\infty}\left(k'_n-\frac{k'_\infty}{n}\right)\frac{\partial P_n(t)}{\partial\psi} \quad (23)$$

⑥水平负荷格林函数$G^l$：

$$G^l(\psi)=-\frac{a}{M}\frac{l'_\infty cos\frac{\psi}{2}\left(1+2sin\frac{\psi}{2}\right)}{2sin\frac{\psi}{2}\left(1+sin\frac{\psi}{2}\right)}+\frac{a}{M}\sum_{n=1}^{\infty}\left(l'_n-\frac{l'_\infty}{n}\right)\frac{\partial P_n(t)}{\partial\psi} \quad (24)$$

⑦径向负荷格林函数$G^r$：

$$G^r(\psi)=\frac{a}{M}\frac{h'_\infty}{2sin\frac{\psi}{2}}+\frac{a}{M}\sum_{n=0}^{\infty}(h'_n-h'_\infty)P_n(t) \quad (25)$$

⑧扰动重力梯度负荷格林函数$G^{T_{nn}}$:

$$G^{T_{nn}}(\psi)=-\frac{\gamma}{aM}\sum_{n=0}^{\infty}(n+1)(n+2)k_n'P_n(t) \tag{26}$$

⑨水平重力梯度间接影响格林函数$G^{T_{ss}}$:

$$G_i^{T_{ss}}(\psi)=\frac{\gamma}{aM}\sum_{n=0}^{\infty}k_n'\frac{\partial^2 P_n(t)}{\partial\psi^2} \tag{27}$$

令$\mathcal{G}_i(l)=2asin\frac{\psi}{2}G_i(\psi)=lG_i(\psi)$，将负荷勒夫数代入式(19)~(27)，求得地表单位点质量负荷$q_w=1$（kg/m$^2$）作用下，各种地面大地测量参数负荷间接影响格林函数值，如表1。

**表1 负荷间接影响格林函数取值**

| $l$(km) | $\mathcal{G}_i^{\zeta}\times10^{-13}$ | $\mathcal{G}_i^{g}\times10^{-17}$ | $\mathcal{G}_i^{\delta g}\times10^{-18}$ | $\mathcal{G}_i^{t}\times10^{-14}$ | $\mathcal{G}_i^{\Theta}\times10^{-19}$ | $\mathcal{G}^{l}\times10^{-12}$ | $\mathcal{G}^{r}\times10^{-11}$ | $\mathcal{G}_i^{nn}\times10^{-15}$ | $\mathcal{G}_i^{ss}\times10^{-15}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | -0.0249 | -11.3315 | 15.8795 | 42.2955 | -2.1192 | -0.8369 | -42.1264 | 40.7525 | 20.0337 |
| 0.2 | -0.0439 | -9.8972 | 29.6981 | 21.1510 | -8.0632 | -3.1842 | -41.9553 | 73.6102 | 34.1831 |
| 0.3 | -0.0625 | -8.8334 | 39.7946 | 14.1058 | -16.6878 | -6.5901 | -41.7788 | 92.3770 | 37.9744 |
| 0.4 | -0.0804 | -8.2348 | 45.2182 | 10.5853 | -26.3601 | -10.4097 | -41.5956 | 93.8712 | 29.4189 |
| 0.5 | -0.0975 | -8.1095 | 45.8894 | 8.4739 | -35.3064 | -13.9425 | -41.4057 | 78.5612 | 9.4993 |
| 0.6 | -0.1139 | -8.3807 | 42.5773 | 7.0657 | -41.9834 | -16.5790 | -41.2101 | 50.3867 | -18.0490 |
| 0.7 | -0.1294 | -8.9073 | 36.7009 | 6.0583 | -45.3905 | -17.9241 | -41.0109 | 15.8142 | -47.6055 |
| 0.8 | -0.1444 | -9.5157 | 30.0034 | 5.3006 | -45.2558 | -17.8704 | -40.8109 | -17.6468 | -72.9744 |
| 1.0 | -0.1727 | -10.3454 | 20.4992 | 4.2343 | -36.8762 | -14.5596 | -40.4173 | -55.8494 | -91.9157 |
| 1.2 | -0.1998 | -10.1321 | 21.4749 | 3.5210 | -26.2416 | -10.3574 | -40.0402 | -39.6641 | -61.0517 |
| 1.4 | -0.2261 | -9.1669 | 30.0077 | 3.0153 | -22.8895 | -9.0304 | -39.6752 | 8.4433 | -7.5471 |
| 1.6 | -0.2518 | -8.3519 | 37.0350 | 2.6419 | -28.6871 | -11.3158 | -39.3091 | 42.4515 | 24.9158 |
| 2.0 | -0.3003 | -8.9633 | 28.5858 | 2.1198 | -40.5309 | -15.9830 | -38.5476 | -4.3817 | -24.2022 |
| 2.5 | -0.3570 | -9.1242 | 24.1119 | 1.6843 | -25.9871 | -10.2232 | -37.6133 | -17.0612 | -27.2278 |
| 3.0 | -0.4112 | -7.9718 | 32.8632 | 1.4080 | -35.2424 | -13.8576 | -36.7093 | 28.7167 | 17.2271 |
| 3.5 | -0.4621 | -8.9437 | 20.3140 | 1.2022 | -32.5321 | -12.7629 | -35.7866 | -31.1746 | -40.2655 |
| 4.0 | -0.5112 | -7.7218 | 29.8481 | 1.0465 | -28.2814 | -11.0562 | -34.9109 | 22.8507 | 15.9355 |
| 5.0 | -0.6036 | -7.8959 | 22.7679 | 0.8291 | -26.3578 | -10.2305 | -33.1702 | -5.9459 | -11.1019 |
| 6.0 | -0.6903 | -7.8527 | 18.1028 | 0.6858 | -29.9324 | -11.5649 | -31.5082 | -23.6048 | -28.4842 |
| 7.0 | -0.7725 | -7.2943 | 18.8748 | 0.5827 | -33.7803 | -12.9988 | -29.9389 | -13.5281 | -18.2480 |
| 8.0 | -0.8510 | -6.5206 | 22.0921 | 0.5013 | -33.1161 | -12.6452 | -28.4652 | 9.3638 | 5.3150 |
| 10.0 | -0.9991 | -6.0125 | 18.9937 | 0.3784 | -24.7530 | -9.1540 | -25.7982 | 5.3162 | 2.8950 |
| 12.0 | -1.1387 | -5.9045 | 13.1167 | 0.2999 | -27.9718 | -10.2454 | -23.5296 | -16.1892 | -18.4692 |
| 14.0 | -1.2726 | -4.9048 | 17.3988 | 0.2398 | -26.5722 | -9.5373 | -21.6664 | 13.0654 | 11.2087 |
| 16.0 | -1.4019 | -4.8896 | 12.8941 | 0.1911 | -21.0009 | -7.2164 | -20.1480 | -4.3047 | -5.5888 |
| 20.0 | -1.6520 | -4.0437 | 14.8205 | 0.1306 | -20.9145 | -7.0582 | -18.0179 | 12.2601 | 11.2369 |
| 25.0 | -1.9534 | -3.6904 | 13.7959 | 0.0872 | -19.8016 | -6.6584 | -16.5317 | 10.0949 | 9.3198 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 30.0 | -2.2455 | -3.5544 | 12.9067 | 0.0638 | -18.9897 | -6.5141 | -15.7982 | 5.5325 | 4.9129 |
| 35.0 | -2.5296 | -3.5250 | 12.0811 | 0.0505 | -18.1729 | -6.4230 | -15.4331 | 0.0753 | -0.4331 |
| 40.0 | -2.8059 | -3.5272 | 11.4345 | 0.0423 | -17.1945 | -6.2698 | -15.2297 | -4.7358 | -5.1568 |
| 50.0 | -3.3365 | -3.4643 | 11.2395 | 0.0322 | -14.9772 | -5.7725 | -14.9607 | -8.1685 | -8.4622 |
| 60.0 | -3.8395 | -3.2518 | 12.5464 | 0.0262 | -13.6029 | -5.4612 | -14.6941 | -2.7549 | -2.9775 |
| 70.0 | -4.3177 | -3.0073 | 14.0654 | 0.0229 | -13.9783 | -5.7205 | -14.3923 | 4.6469 | 4.4506 |
| 80.0 | -4.7741 | -2.8804 | 14.3310 | 0.0210 | -15.3999 | -6.3101 | -14.0649 | 6.2127 | 6.0235 |
| 100.0 | -5.6311 | -2.9117 | 11.9306 | 0.0171 | -15.7804 | -6.3810 | -13.3843 | -4.6763 | -4.8316 |
| 120.0 | -6.4270 | -2.6545 | 12.4755 | 0.0129 | -14.0249 | -5.5346 | -12.7235 | 0.1761 | 0.0607 |
| 140.0 | -7.1738 | -2.4359 | 12.7461 | 0.0120 | -15.5946 | -5.9880 | -12.0989 | 3.7448 | 3.6348 |
| 160.0 | -7.8804 | -2.4586 | 10.7233 | 0.0100 | -14.9953 | -5.5941 | -11.5133 | -4.4893 | -4.5820 |
| 180.0 | -8.5536 | -2.2087 | 11.5710 | 0.0080 | -13.8312 | -4.9933 | -10.9748 | 1.9062 | 1.8299 |
| 200.0 | -9.1986 | -2.0952 | 11.1758 | 0.0080 | -15.1075 | -5.3733 | -10.4758 | 1.7439 | 1.6689 |
| 250.0 | -10.7136 | -1.8097 | 10.7082 | 0.0058 | -14.0435 | -4.7072 | -9.3924 | 3.2869 | 3.2307 |
| 300.0 | -12.1238 | -1.5962 | 10.1419 | 0.0042 | -12.9077 | -4.0819 | -8.5118 | 3.2916 | 3.2481 |
| 350.0 | -13.4587 | -1.4397 | 9.5227 | 0.0030 | -11.9089 | -3.5581 | -7.7994 | 2.1184 | 2.0836 |
| 400.0 | -14.7375 | -1.3210 | 8.9521 | 0.0023 | -11.1503 | -3.1625 | -7.2265 | 0.4258 | 0.3969 |
| 500.0 | -17.1749 | -1.1331 | 8.3207 | 0.0016 | -10.3019 | -2.7029 | -6.4078 | -2.1612 | -2.1831 |
| 600.0 | -19.4980 | -0.9603 | 8.5053 | 0.0014 | -9.8691 | -2.4641 | -5.9044 | -2.3040 | -2.3219 |
| 700.0 | -21.7353 | -0.8020 | 9.1773 | 0.0012 | -9.4436 | -2.2596 | -5.6076 | -1.0615 | -1.0768 |
| 800.0 | -23.8986 | -0.6720 | 9.9646 | 0.0010 | -9.0007 | -2.0628 | -5.4405 | 0.1041 | 0.0908 |

类似基于全球重力场模位系数模型的局部重力场精化作业流程，区域负荷形变场精化，可采用基于全球负荷形变场球谐系数模型的移去恢复法。

**3、勒让德函数$P_n(cos\,\psi)$及其对$\psi$一、二阶导数**

令$t = cos\,\psi$，　$u = sin\,\psi$　　(26)

$$P_n(t) = \frac{2n-1}{n} t P_{n-1}(t) - \frac{n-1}{n} P_{n-2}(t) \quad (27)$$

$$P_1 = t, \quad P_2 = \frac{1}{2}(3t^2 - 1) \quad (28)$$

$$\frac{\partial}{\partial \psi} P_n(t) = \frac{2n-1}{n} t \frac{\partial}{\partial \psi} P_{n-1}(t) - \frac{2n-1}{n} u P_{n-1}(t) - \frac{n-1}{n} \frac{\partial}{\partial \psi} P_{n-2}(t) \quad (29)$$

$$\frac{\partial}{\partial \psi} P_1(t) = -u, \quad \frac{\partial}{\partial \psi} P_2(t) = -3ut \quad (30)$$

$$\frac{\partial^2}{\partial \psi^2} P_n(t) = \frac{2n-1}{n} \left( t \frac{\partial^2}{\partial \psi^2} P_{n-1} - 2u \frac{\partial}{\partial \psi} P_{n-1} - t P_{n-1} \right) - \frac{n-1}{n} \frac{\partial^2}{\partial \psi^2} P_{n-2} \quad (31)$$

$$\frac{\partial^2}{\partial \psi^2} P_1(t) = -t, \quad \frac{\partial^2}{\partial \psi^2} P_2(t) = 3(1 - 2t^2) \quad (32)$$