## Integral algorithm formula of anomalous gravity field

### 7.9.1 Stokes and Hotine integral formulas outside geoid

(1) It is known that the gravity anomaly $\Delta g$ on an equipotential surface outside the geoid, the disturbing potential $T(r,\theta,\lambda)$ or the height anomaly $\zeta(r,\theta,\lambda)$ at the calculation point $(r,\theta,\lambda)$ outside the geoid can be calculated by Stokes integral Formula:

$$T(r,\theta,\lambda)=\gamma\zeta(r,\theta,\lambda)=\frac{1}{4\pi}\iint_s \Delta g' S(r,\psi,r')ds \tag{9.1}$$

where $r'$ is the geocentric distance of the move areal element $ds$ (the move point) on the equipotential surface where the gravity anomaly $\Delta g'$ is located, $S(r,\psi,r')$ is called as the generalized Stokes kernel function, and:

$$S(r,\psi,r')=\frac{2}{L}+\frac{1}{r}-\frac{3L}{r^2}-\frac{5r'cos\psi}{r^2}-\frac{3r'}{r^2}cos\psi ln\frac{r-r'cos\psi+L}{2r} \tag{9.2}$$

where $L$ is the space distance from the move point to the calculation point.

When the calculation point is the same as the move point, the integral is singular:

$$\zeta|_0=\frac{A_0}{\gamma}\Delta g_0 \tag{9.3}$$

(2) It is known that the gravity disturbance $\delta g$ on an equipotential surface outside the geoid, the disturbing potential $T(r,\theta,\lambda)$ or the height anomaly $\zeta(r,\theta,\lambda)$ at the calculation point $(r,\theta,\lambda)$ outside the geoid can be calculated by Hotine integral Formula:

$$T(r,\theta,\lambda)=\gamma\zeta(r,\theta,\lambda)=\frac{1}{4\pi}\iint_s \delta g' H(r,\psi,r')ds \tag{9.4}$$

where $H(r,\psi,r')$ is called as the generalized Hotine kernel function, and:

$$H(r,\psi,r')=\frac{2}{L}-\frac{1}{r}-\frac{3r'cos\psi}{r^2}-\frac{1}{r'}ln\frac{r-r'cos\psi+L}{r(1-cos\psi)} \tag{9.5}$$

When the calculation point is the same as the move point, the integral is singular:

$$\zeta|_0=\frac{A_0}{\gamma}\delta g_0 \tag{9.6}$$

If $r$ and $r'$ are taken as constants, the generalized Stokes/Hotine integral can be calculated by the fast FFT algorithm.

### 7.9.2 Vening-Meinesz integral formulas outside geoid

Taking the horizontal derivatives on both sides of the generalized Stokes formula, we have:

$$\xi=\frac{-1}{4\pi r\gamma}\iint_s \Delta g'\frac{\partial S(r,\psi,r')}{\partial\psi}\frac{\partial\psi}{\partial\varphi}ds,\quad \eta=\frac{-1}{4\pi r cos\varphi\gamma}\iint_s \Delta g'\frac{\partial S(r,\psi,r')}{\partial\psi}\frac{\partial\psi}{\partial\lambda}ds \tag{9.7}$$

From $cos\psi=sin\varphi sin\varphi'+cos\varphi cos\varphi' cos(\lambda'-\lambda)$, (9.8)

differentiating both sides, we get:

$$-sin\psi\frac{\partial\psi}{\partial\varphi}=cos\varphi sin\varphi'-sin\varphi cos\varphi' cos(\lambda'-\lambda) \tag{9.9}$$

$$-sin\psi\frac{\partial\psi}{\partial\lambda}=cos\varphi cos\varphi' sin(\lambda'-\lambda) \tag{9.10}$$

From the spherical trigonometry formula, we can get:

$$sin\psi cos\alpha = cos\varphi sin\varphi' - sin\varphi cos\varphi' cos(\lambda' - \lambda) \quad (9.11)$$

$$sin\psi sin\alpha = cos\varphi' sin(\lambda' - \lambda) \quad (9.12)$$

Combining formulas (9.9) ~ (9.12), we can get:

$$\frac{\partial\psi}{\partial\varphi} = -cos\alpha, \quad \frac{\partial\psi}{\partial\lambda} = -cos\varphi sin\alpha \quad (9.13)$$

Substitute formula (9.13) into (9.7), we have:

$$\xi = \frac{1}{4\pi r\gamma}\iint_s g' \frac{\partial S(r,\psi,r')}{\partial\psi} cos\alpha ds, \quad \eta = \frac{1}{4\pi r\gamma}\iint_s \Delta g' \frac{\partial S(r,\psi,r')}{\partial\psi} sin\alpha ds \quad (9.14)$$

Considering $L = \sqrt{r^2 + r'^2 - 2rr'cos\psi}$, we have:

$$\frac{\partial}{\partial\psi}L = \frac{rr'}{L}sin\psi, \quad \frac{\partial}{\partial\psi}\frac{1}{L} = -\frac{1}{L^2}\frac{\partial}{\partial\psi}L = -\frac{rr'}{L^3}sin\psi \quad (9.15)$$

$$\frac{\partial}{\partial\psi}ln\frac{r-r'cos\psi+L}{2r} = \frac{1}{r-r'cos\psi+L}\left(\frac{rr'}{L}sin\psi + r'sin\psi\right) = \frac{r'sin\psi}{r+L-r'cos\psi}\frac{L+r}{L} \quad (9.16)$$

$$\frac{\partial}{\partial\psi}S(r,\psi,r') = \frac{\partial}{\partial\psi}\left(\frac{2}{L} + \frac{1}{r} - \frac{3L}{r^2} - \frac{5r'cos\psi}{r^2} - \frac{3r'cos\psi}{r^2}ln\frac{r-r'cos\psi+L}{2r}\right)$$

$$= \frac{\partial}{\partial\psi}\frac{2}{L} - \frac{3}{r^2}\frac{\partial}{\partial\psi}L + \frac{5r'sin\psi}{r^2} + \frac{3r'sin\psi}{r^2}ln\frac{r+L-r'cos\psi}{2r} - \frac{3r'cos\psi}{r^2}\frac{\partial}{\partial\psi}ln\frac{r+L-r'cos\psi}{2r}$$

$$= \left(-\frac{2rr'}{L^3} - \frac{3r'}{rL} + \frac{5r'}{r^2} + \frac{3r'}{r^2}ln\frac{r-r'cos\psi+L}{2r} - \frac{3r'cos\psi}{r^2}\frac{r'}{r-r'cos\psi+L}\frac{L+r}{L}\right)sin\psi$$

$$= \left[-\frac{2r}{L^3} - \frac{3}{rL} + \frac{5}{r^2} + \frac{3}{r^2}ln\frac{r-r'cos\psi+L}{2r} - \frac{3r'(L+r)cos\psi}{r^2L(r-r'cos\psi+L)}\right]r'sin\psi \quad (9.17)$$

In the same way, by calculating the horizontal derivatives on both sides of the generalized Hotine formula, we can get:

$$\xi = \frac{1}{4\pi r\gamma}\iint_s \delta g' \frac{\partial H(r,\psi,r')}{\partial\psi} cos\alpha ds, \quad \eta = \frac{1}{4\pi r\gamma}\iint_s \delta g' \frac{\partial H(r,\psi,r')}{\partial\psi} sin\alpha ds \quad (9.18)$$

Because of

$$\frac{\partial}{\partial\psi}ln\frac{r-r'cos\psi+L}{r(1-cos\psi)} = \frac{r(1-cos\psi)}{r-r'cos\psi+L}\frac{\left(\frac{rr'}{L}sin\psi + r'sin\psi\right)r(1-cos\psi) + (r-r'cos\psi+L)rsin\psi}{r^2(1-cos\psi)^2}$$

$$= \frac{sin\psi}{r-r'cos\psi+L}\frac{\frac{L+r}{L}r'(1-cos\psi) + (r-r'cos\psi+L)}{1-cos\psi} = \left[\frac{r'(L+r)}{(r-r'cos\psi+L)L} + \frac{1}{1-cos\psi}\right]sin\psi, \quad (9.19)$$

we have:

$$\frac{\partial}{\partial\psi}H(r,\psi,r') = \frac{\partial}{\partial\psi}\left(\frac{2}{L} - \frac{1}{r} - \frac{3r'cos\psi}{r^2} - \frac{1}{r'}ln\frac{r-r'cos\psi+L}{r(1-cos\psi)}\right)$$

$$= \frac{\partial}{\partial\psi}\frac{2}{L} + \frac{3r'sin\psi}{r^2} - \frac{1}{r'}\frac{\partial}{\partial\psi}ln\frac{r-r'cos\psi+L}{r(1-cos\psi)}$$

$$= \left[-\frac{2rr'}{L^3} + \frac{3r'}{r^2} - \frac{L-r}{(r-r'cos\psi+L)L} + \frac{1}{r'(1-cos\psi)}\right]sin\psi \quad (9.20)$$

Formulas (9.14) and (9.18) are also called as generalized Vening-Meinesz integral formulas, and formulas (9.17) and (9.20) are generalized Vening-Meinesz kernel functions.

Using the formula (9.14), the vertical deflection at any point outside the geoid be calculated from the gravity anomaly on a certain equipotential surface. And using the formula (9.18), the vertical deflection at any point outside the geoid be calculated from the gravity disturbance on a certain equipotential surface.

If $r$ and $r'$ are taken as constants, the generalized Vening-Meinesz integral formulas (9.14) and (9.18) can be calculated by the fast FFT algorithm.

### 7.9.3 Integral formula of inverse operation of anomalous gravity field element

(1) Calculation of the gravity disturbance by integral on the height anomaly

According to the definition of gravity disturbance, take the vertical derivative of the Poisson integral formula of disturbing potential $T$

$$\delta g = \frac{\partial T}{\partial n} \approx -\frac{\gamma \partial \zeta}{\partial r} = -\frac{\gamma}{2\pi} \iint_{s} \frac{\zeta - \zeta_p}{l^3} ds \tag{9.21}$$

where $n$ is the vertical line direction (reverse to the radial direction $r$), and $l$ is the distance between the calculation point and the move point on the sphere:

$$l = 2r \sin\frac{\psi}{2} \tag{9.22}$$

Formula (9.21) is also known as the inverse Hotine integral formula under spherical approximation.

When the calculation point is the same as the move point, the integral is singular:

$$\delta g|_0 = \frac{\gamma\sqrt{A_0/\pi}}{4} \left(\zeta_{xx} + \zeta_{yy}\right) \tag{9.23}$$

where $\zeta_{xx}$ and $\zeta_{yy}$ are the second-order horizontal partial derivatives of the height anomaly at the calculation point, and $\gamma\zeta_{xx}$ and $\gamma\zeta_{yy}$ are the northward direction of the horizontal gravity gradient and the eastward direction of the horizontal gravity gradient, respectively.

Using formula (9.21), the gravity disturbance on the equipotential surface can be calculated from the height anomaly on the surface.

Since the gravity disturbance $\delta g$ is the derivative of the disturbing potential $T$ along the vertical direction n, formular (9.21) requires that the boundary surface where the height anomaly is located should be an equipotential surface.

(2) Calculation of the gravity anomaly by integral on the height anomaly

Substitute the basic gravimetric equation into formular (9.21) to get:

$$\Delta g = -\frac{\gamma}{2\pi} \iint_{s} \frac{\zeta - \zeta_p}{l^3} ds - \frac{\zeta\gamma}{2r} \tag{9.24}$$

Formula (9.24) is also known as the inverse Stokes integral formula under spherical approximation.

Using formula (9.24), the gravity anomaly on the equipotential surface can be calculated from the height anomaly on the surface.

(3) Calculation of the height anomaly by integral on the vertical deflection

$$\zeta = \frac{r}{4\pi} \iint_{\sigma} ctg\frac{\psi}{2} (\xi cos\alpha + \eta sin\alpha)\, d\sigma \tag{9.25}$$

When the calculation point is the same as the move point, the integral is singular:

$$\zeta|_0 = \frac{A_0}{4\pi} \left(\xi_y + \eta_x\right) \tag{9.26}$$

where $\xi_y$ and $\eta_x$ are the partial derivatives of $\xi$ and $\eta$ in the east and north directions, respectively.

Using formula (9.26), the height anomaly on the equipotential surface can be calculated from the vertical deflection on the surface.

(4) Calculation of the gravity anomaly by integral on the vertical deflection

$$\Delta g = -\frac{\gamma}{4\pi}\iint_{\sigma}\left(3csc\psi - csc\psi csc\frac{\psi}{2} - tg\frac{\psi}{2}\right)(\xi cos\alpha + \eta sin\alpha)\,d\sigma \qquad (9.27)$$

When the calculation point is the same as the move point, the integral is singular:

$$\Delta g|_0 = -\frac{\gamma\sqrt{A_0/\pi}}{4}\left(\xi_y + \eta_x\right) \qquad (9.28)$$

Using formula (9.27), the gravity anomaly on the equipotential surface can be calculated from the vertical deflection on the surface.

(5) Calculation of the gravity disturbance by integrating on the vertical deflection

From the basic gravimetric equation, and the formulas (9.25) and (9.27), the integral formula for the gravity disturbance from the vertical deflection can be obtained:

$$\delta g = \frac{-\gamma}{4\pi}\iint_{\sigma}\left(3csc\psi - csc\psi csc\frac{\psi}{2} - tg\frac{\psi}{2} - 2ctg\frac{\psi}{2}\right)(\xi cos\alpha + \eta sin\alpha)\,d\sigma \qquad (9.29)$$

When the calculation point is the same as the move point, the integral is singular:

$$\delta g|_0 = -\frac{\gamma}{2\pi}\left(\sqrt{\pi A_0} + \frac{A_0}{r}\right)\left(\xi_y + \eta_x\right) \qquad (9.30)$$

Using formula (9.29), the gravity disturbance on the equipotential surface can be calculated from the vertical deflection on the surface.

Formulas (9.25), (9.27) and (9.29) are also known as the inverse Vening-Meinesz integral formula under spherical approximation.

If $r$ is taken as constant, formulas (9.21), (9.24), (9.25), (9.27) and (9.29) can all be calculated by the fast FFT algorithm.

### 7.9.4 Positive and negative operation formula of anomalous field element integral

(1) Poisson integral formula of anomalous field element

Any type of anomalous gravity field element $\mu$ can be expressed by the linear combination of the disturbing potential or its partial derivatives. Therefore, the radial gradient and Poisson integral formula of field element are similar.

Given the anomalous gravity field element on a certain boundary surface, the Poisson integral relation satisfied by the same type of field element at any point $(r,\theta,\lambda)$ outside the geoid:

$$\mu(r,\theta,\lambda) = \frac{1}{4\pi r}\iint_{S}\mu'\frac{r^2 - r'^2}{L^3}\,ds \qquad (9.31)$$

When the calculation point is the same as the move point, $\psi \to 0$, $r' \to rt$, $L \to r\psi$ and $r - r't \to r\psi^2$, the integral is singular. Considering

$$ds = r'^2 sin\psi d\psi d\alpha = \pi r^2\psi_0^2 \qquad (9.32)$$

we have $\frac{1}{4\pi r}\iint_s \frac{r^2-r'^2}{L^3}ds=\frac{1}{2r}\int_0^{\psi_0} r^2\frac{\psi^2}{r^3\psi^3}r^2\psi d\psi=\frac{1}{2}\psi_0=\frac{1}{2r}\sqrt{ds/\pi}$, (9.33)

hence $\mu|_0=\frac{\mu'}{2r}\sqrt{ds/\pi}$. (9.34)

(2) Radial gradient integral formula for anomalous field elements

Given the anomalous gravity field element on a certain equipotential surface, the radial gradient of the field element in the Stokes boundary value theory can be calculated by the following integral formula:

$$\frac{\partial\mu}{\partial r}=\frac{1}{2\pi}\iint_s \frac{\mu-\mu'}{l^3}ds \quad (9.35)$$

If $r$ and $r'$ are taken as constants, the integral formulas (9.31) and (9.35) can be calculated by the fast FFT algorithm.

(3) Integral positive and negative operation formula for disturbing gravity gradient

Given the disturbing gravity gradient $T_{rr}$ on some an equipotential surface outside the geoid, the gravity disturbance $\delta g=-T_r$ at any calculation point $(r,\theta,\lambda)$ outside the geoid satisfies the following integral formula:

$$\delta g(r,\theta,\lambda)=\frac{1}{4\pi}\iint_s T_{rr}H(r,\psi,r')ds \quad (9.36)$$

where $H(r,\psi,r')$ is the generalized Hotine kernel function.

Given the gravity disturbance $\delta g$ on a certain equipotential surface, the disturbing gravity gradient at any point on the equipotential surface can be calculated by the following integral formula:

$$T_{rr}=\frac{1}{2\pi}\iint \frac{\delta g-\delta g'}{l^3}ds \quad (9.37)$$

(4) Calculation of the disturbing gravity gradient by integral on the gravity disturbance

Given the gravity disturbance $\delta g$ on a certain boundary surface, the disturbing gravity gradient $T_{rr}$ at the any point $(r,\theta,\lambda)$ outside the geoid can also be calculated.

Using the Poisson integral formular (9.31) for the gravity disturbance $\delta g$, we have:

$$\delta g(r,\theta,\lambda)=\frac{1}{4\pi r}\iint_s \delta g'\frac{r^2-r'^2}{L^3}ds \quad (9.38)$$

Considering $T_{rr}=\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}T\right)=-\frac{\partial}{\partial r}(\delta g)$, taking the partial derivatives of both sides of (9.38) with respect to $r$, we get:

$$T_{rr}=-\frac{1}{4\pi r}\iint_s \delta g'\frac{\partial}{\partial r}\frac{r^2-r'^2}{L^3}ds=\frac{1}{4\pi r}\iint_s \delta g'\frac{r^3-5rr'^2+(r^2+3r'^2)r'^2\cos\psi}{L^5}ds \quad (9.39)$$

The formula (9.39) for calculating the disturbing gravity gradient outside the geoid from the gravity disturbance on the boundary surface is derived from the Poisson integral formula for solving the first boundary value problem. Therefore, it is not required that the boundary surface should be a gravity equipotential surface.

**Calculation formulas of normal gravity field at any point in Earth space**

(1) The normal geopotential U at the Earth space point $(r,\theta,\lambda)$ in the spherical coordinate system can be expressed as spherical harmonic series:

$$U(r,\theta)=\frac{GM}{r}\left[1-\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{2n}J_{2n}P_{2n}(\cos\theta)\right]+\frac{1}{2}\omega^2r^2\sin^2\theta \quad (1.1)$$

$$J_{2n}=(-1)^{n+1}\frac{3e^{2n}}{(2n+1)(2n+3)}\left(1-n+\frac{5nJ_2}{e^2}\right) \quad (1.2)$$

where $r$ is the distance from the calculation point to the center of the level ellipsoid, $\lambda$ is the longitude of the calculation point, $\theta=\pi/2-\varphi$ is the geocentric colatitude, $\varphi$ is the geocentric latitude, $a$ is the semimajor axis of the ellipsoid, $J_2$ is the dynamic shape factor of the Earth, $GM$ is the geocentric gravitational constant, $\omega$ is the mean rotation rate of the Earth, $e$ is the first eccentricity of the level ellipsoid, and $P_{2n}(cos\theta)$ is the Legendre function.

(2) Taking the partial derivative of the normal gravitational potential $U(r,\theta)$ formula (1.1) in the spherical coordinate system, the normal gravity vector at the Earth space point can be obtained:

$$\gamma(r,\theta)=\gamma_r e_r+\gamma_\theta e_\theta \quad (1.3)$$

$$\gamma_r=-\frac{GM}{r^2}\left[1-\sum_{n=1}^{\infty}(2n+1)\left(\frac{a}{r}\right)^{2n}J_{2n}P_{2n}(cos\theta)\right]+\omega^2 rsin^2\theta \quad (1.4)$$

$$\gamma_\theta=\frac{\partial U}{r\partial\theta}=-\frac{GM}{r^2}\left[\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{2n}J_{2n}\frac{\partial}{\partial\theta}P_{2n}(cos\,\theta)\right]+\omega^2 r\,sin\,\theta\,cos\theta \quad (1.5)$$

Since $e_r\perp e_\theta$, the normal gravity scalar value can be obtained:

$$\gamma=\sqrt{\gamma_r^2+\gamma_\theta^2} \quad (1.6)$$

and the north declination angle of the normal gravity line direction relative to the Earth center of mass can also be obtained:

$$\vartheta_\gamma=tan^{-1}\frac{\gamma_\theta}{\gamma_r} \quad (1.7)$$

(3) Furtherly taking the partial derivative of the normal gravity vector $\gamma(r,\theta)$ of the formula (1.3), the diagonal elements of the normal gravity gradient tensor at the earth space point in the spherical coordinate system can be obtained:

$$U_{rr}=-2\frac{GM}{r^3}\left[1-\sum_{n=1}^{\infty}(n+1)(2n+1)\left(\frac{a}{r}\right)^{2n}J_{2n}P_{2n}(cos\theta)\right]+\omega^2 sin^2\theta \quad (1.8)$$

$$U_{\theta\theta}=\frac{\partial^2 U}{r^2\partial\theta^2}=\frac{\partial\gamma_\theta}{r\partial\theta}=-\frac{GM}{r^3}\left[\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{2n}J_{2n}\frac{\partial^2}{\partial\theta^2}P_{2n}(cos\,\theta)\right]+\omega^2\,cos2\,\theta \quad (1.9)$$

Since $e_r\perp e_\theta$, the normal gravity gradient scalar value can be obtained:

$$U_{nn}=\sqrt{U_{rr}^2+U_{\theta\theta}^2} \quad (1.10)$$

and the north declination angle of the normal gravity gradient direction relative to the mass center of the Earth can also be obtained:

$$\vartheta_E=tan^{-1}\frac{U_{\theta\theta}}{U_{rr}} \quad (1.11)$$

(4) Low-dgree Legendre function $P_n(t)$ and its first and second derivative algorithms with respect to $\theta$

Let $t = \cos\theta, \quad u = \sin\theta,$ (1.12)

then $P_n(t) = \frac{2n-1}{n} tP_{n-1}(t) - \frac{n-1}{n} P_{n-2}(t)$ (1.13)

$$P_1 = t, \quad P_2 = \frac{1}{2}(3t^2 - 1) \tag{1.14}$$

$$\frac{\partial}{\partial\theta} P_n(t) = \frac{2n-1}{n} t \frac{\partial}{\partial\theta} P_{n-1}(t) - \frac{2n-1}{n} uP_{n-1}(t) - \frac{n-1}{n} \frac{\partial}{\partial\theta} P_{n-2}(t) \tag{1.15}$$

$$\frac{\partial}{\partial\theta} P_1(t) = -u, \quad \frac{\partial}{\partial\theta} P_2(t) = -3ut \tag{1.16}$$

$$\frac{\partial^2}{\partial\theta^2} P_n(t) = \frac{2n-1}{n}\left(t\frac{\partial^2}{\partial\theta^2} P_{n-1} - 2u\frac{\partial}{\partial\theta} P_{n-1} - tP_{n-1}\right) - \frac{n-1}{n}\frac{\partial^2}{\partial\theta^2} P_{n-2} \tag{1.17}$$

$$\frac{\partial^2}{\partial\theta^2} P_1(t) = -t, \quad \frac{\partial^2}{\partial\theta^2} P_2(t) = 3(1 - 2t^2) \tag{1.18}$$

**Calculation formulas of Earth gravity field from geopotential coefficient model**

The disturbing potential $T$ or height anomaly $\zeta$ at the space point $(r, \theta, \lambda)$ outside the Earth can be expressed as the following spherical harmonic series:

$$T(r,\theta,\lambda) = \zeta\gamma = \frac{GM}{r}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}(\delta\bar{C}_{nm}cosm\lambda + \bar{S}_{nm}sinm\lambda)\bar{P}_{nm} \tag{2.1}$$

where $\bar{C}_{nm}, \bar{S}_{nm}$ are called as the fully normalized Stokes coefficients, also known as the geopotential coefficients, $\bar{P}_{nm} = \bar{P}_{nm}(t)$ is the fully normalized associative Legendre function, n is called as the degree of the geopotential coefficient, m is called as order of geopotential coefficients. And

$$\delta\bar{C}_{2n,0} = \bar{C}_{2n,0} + \frac{J_{2n}}{\sqrt{4n+1}} \tag{2.2}$$

$$\delta\bar{C}_{2n,m} = \bar{C}_{2n,m}(m > 0) \quad \delta\bar{C}_{2n+1,m} = \bar{C}_{2n+1,m} \tag{2.3}$$

The spherical harmonic series of gravity anomaly $\Delta g$, gravity disturbance $\delta g$, vertical deflection $(\xi, \eta)$, disturbing gravity gradient $T_{rr}$ and tangential gravity gradient $(T_{NN}, T_{WW})$ at the space point $(r, \theta, \lambda)$ outside the Earth can be respectively expressed as:

$$\Delta g = \frac{GM}{r^2}\sum_{n=2}^{\infty}(n-1)\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}(\delta\bar{C}_{nm}cosm\lambda + \bar{S}_{nm}sinm\lambda)\bar{P}_{nm} \tag{2.4}$$

$$\delta g = -T_r = \frac{GM}{r^2}\sum_{n=2}^{\infty}(n+1)\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}(\delta\bar{C}_{nm}cosm\lambda + \bar{S}_{nm}sinm\lambda)\bar{P}_{nm} \tag{2.5}$$

$$\xi = \frac{T_\theta}{\gamma r} = \frac{GM}{\gamma r^2}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}(\delta\bar{C}_{nm}cosm\lambda + \bar{S}_{nm}sinm\lambda)\frac{\partial}{\partial\theta}\bar{P}_{nm} \tag{2.6}$$

$$\eta = -\frac{T_\lambda}{\gamma r sin\theta} = \frac{GM}{\gamma r^2 sin\theta}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=1}^{n} m(\delta\bar{C}_{nm}sinm\lambda - \bar{S}_{nm}cosm\lambda)\bar{P}_{nm} \tag{2.7}$$

$$T_{rr} = \frac{GM}{r^3}\sum_{n=2}^{\infty}(n+1)(n+2)\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}(\delta\bar{C}_{nm}cosm\lambda + \bar{S}_{nm}sinm\lambda)\bar{P}_{nm} \tag{2.8}$$

$$T_{NN} = \frac{1}{r}T_r + \frac{1}{r^2}T_{\theta\theta}$$

$$= -\frac{\delta g}{r} + \frac{GM}{r^3}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}(\delta\bar{C}_{nm}cosm\lambda + \bar{S}_{nm}sinm\lambda)\frac{\partial^2}{\partial\theta^2}\bar{P}_{nm} \tag{2.9}$$

$$T_{WW} = \frac{1}{r}T_r + \frac{1}{r^2}T_\theta ctg\theta + \frac{1}{r^2 sin^2\theta}T_{\lambda\lambda} = -\frac{\delta g}{r} + \frac{\xi\gamma}{r}ctg\theta$$

$$-\frac{GM}{r^3 sin^2\theta}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^n\sum_{m=0}^{n} m^2(\delta\bar{C}_{nm}cosm\lambda+\bar{S}_{nm}sinm\lambda)\bar{P}_{nm} \quad (2.10)$$

here, $T_r=\frac{\partial}{\partial r}T(r,\theta,\lambda),\quad T_{rr}=\frac{\partial^2}{\partial r^2}T(r,\theta,\lambda)$ (2.11)

$$T_\theta=\frac{\partial}{\partial\theta}T(r,\theta,\lambda),\quad T_{\theta\theta}=\frac{\partial^2}{\partial\theta^2}T(r,\theta,\lambda) \quad (2.12)$$

$$T_\lambda=\frac{\partial}{\partial\lambda}T(r,\theta,\lambda),\quad T_{\lambda\lambda}=\frac{\partial^2}{\partial\lambda^2}T(r,\theta,\lambda) \quad (2.13)$$

$$T_{rr}+T_{NN}+T_{WW}\equiv 0,\quad T_{rr}^n+T_{NN}^n+T_{NN}^n\equiv 0,\quad T_*=\sum_{n=2}^{\infty}T_*^n \quad (2.14)$$

where $T_*^n$ represents the degree n harmonic component of $T_*$. The N axis points North and the W axis points West.

Equation (2.14) is the Laplace equation, which can be employed to check the spatial and spectral domain performance of the geopotential model.

**Algorithms of normalized associative Legendre function and its derivative**

(1) Standard forward column recursion algorithm for $\bar{P}_{nm}(t)$ (n<1900)

$$\begin{cases}\bar{P}_{nm}(t)=a_{nm}t\bar{P}_{n-1,m}(t)-b_{nm}\bar{P}_{n-2,m}(t) & \forall n>1, m<n\\ \bar{P}_{nn}(t)=u\sqrt{\frac{2n+1}{2n}}\bar{P}_{n-1,n-1} & n>1\end{cases} \quad (3.1)$$

$$a_{nm}=\sqrt{\frac{(2n-1)(2n+1)}{(n+m)(n-m)}},\quad b_{nm}=\sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(2n-3)(n+m)(n-m)}}$$

$$\bar{P}_{00}(t)=1,\quad \bar{P}_{10}(t)=\sqrt{3}t,\quad \bar{P}_{11}(t)=\sqrt{3}u \quad (3.2)$$

(2) Improved Belikov recursion algorithm for $\bar{P}_{nm}(t)$ (n<64800)

When $n=0{,}1$, $\bar{P}_{nm}(t)$ is calculated according to (3.2). And when n≥2, we have:

$$\bar{P}_{n0}(t)=a_n t\bar{P}_{n-1,0}(t)-b_n\frac{u}{2}\bar{P}_{n-1,1}(t),\ m=0 \quad (3.3)$$

$$\bar{P}_{nm}(t)=c_{nm}t\bar{P}_{n-1,m}(t)-d_{nm}u\bar{P}_{n-1,m+1}(t)+e_{nm}u\bar{P}_{n-1,m-1}(t), m>0 \quad (3.4)$$

$$a_n=\sqrt{\frac{2n+1}{2n-1}},\quad b_n=\sqrt{\frac{2(n-1)(2n+1)}{n(2n-1)}} \quad (3.5)$$

$$c_{nm}=\frac{1}{n}\sqrt{\frac{(n+m)(n-m)(2n+1)}{2n-1}},\quad d_{nm}=\frac{1}{2n}\sqrt{\frac{(n-m)(n-m-1)(2n+1)}{2n-1}} \quad (3.6)$$

here when m>0,

$$e_{nm}=\frac{1}{2n}\sqrt{\frac{2}{2-\delta_0^{m-1}}}\sqrt{\frac{(n+m)(n+m-1)(2n+1)}{2n-1}} \quad (3.7)$$

(3) Cross degree-order recursive algorithm for $\bar{P}_{nm}(t)$ (n<20000)

When $n=0{,}1$, $\bar{P}_{nm}(t)$ is calculated according to (3.2). And when n≥2, we have:

$$\bar{P}_{nm}(t)=\alpha_{nm}\bar{P}_{n-2,m}(t)+\beta_{nm}\bar{P}_{n-2,m-2}(t)-\gamma_{nm}\bar{P}_{n,m-2}(t) \quad (3.8)$$

$$\alpha_{nm}=\sqrt{\frac{(2n+1)(n-m)(n-m-1)}{(2n-3)(n+m)(n+m-1)}}$$

$$\beta_{nm}=\sqrt{1+\delta_0^{m-2}}\sqrt{\frac{(2n+1)(n+m-2)(n+m-3)}{(2n-3)(n+m)(n+m-1)}} \quad (3.9)$$

$$\gamma_{nm} = \sqrt{1+\delta_0^{m-2}}\sqrt{\frac{(n-m+1)(n-m+2)}{(n+m)(n+m-1)}}$$

(4) Non-singular recursive algorithm for $\frac{\partial}{\partial\theta}\bar{P}_{nm}(\cos\theta)$

Considering that the first derivative of $\bar{P}_{nm}(\cos\theta)$ with respect to $\theta$ is

$$\frac{\partial}{\partial\theta}\bar{P}_{nm}(\cos\theta) = -\sin\theta\frac{\partial}{\partial t}\bar{P}_{nm}(t) \tag{3.10}$$

we have

$$\begin{cases}\frac{\partial}{\partial\theta}\bar{P}_{n0} = -\sqrt{\frac{n(n+1)}{2}}\bar{P}_{n1}, \quad \frac{\partial}{\partial\theta}\bar{P}_{n1} = \sqrt{\frac{n(n+1)}{2}}\bar{P}_{n0} - \frac{\sqrt{(n-1)(n+2)}}{2}\bar{P}_{n2} \\ \frac{\partial}{\partial\theta}\bar{P}_{nm} = \frac{\sqrt{(n+m)(n-m+1)}}{2}\bar{P}_{n,m-1} - \frac{\sqrt{(n-m)(n+m+1)}}{2}\bar{P}_{n,m+1}, \quad m>2\end{cases} \tag{3.11}$$

$$\frac{\partial}{\partial\theta}\bar{P}_{00}(t) = 0, \quad \frac{\partial}{\partial\theta}\bar{P}_{10}(t) = -\sqrt{3}u, \quad \frac{\partial}{\partial\theta}\bar{P}_{11}(t) = \sqrt{3}t \tag{3.12}$$

(5) Non-singular recursive algorithm for $\frac{\partial^2}{\partial\theta^2}\bar{P}_{nm}(\cos\theta)$

$$\begin{cases}\frac{\partial^2}{\partial\theta^2}\bar{P}_{n0} = -\frac{n(n+1)}{2}\bar{P}_{n0} + \sqrt{\frac{n(n-1)(n+1)(n+2)}{8}}\bar{P}_{n2} \\ \frac{\partial^2}{\partial\theta^2}\bar{P}_{n1} = -\frac{2n(n+1)+(n-1)(n+2)}{4}\bar{P}_{n1} + \frac{\sqrt{(n-2)(n-1)(n+2)(n+3)}}{4}\bar{P}_{n3}\end{cases} \tag{3.13}$$

$$\begin{aligned}\frac{\partial^2}{\partial\theta^2}\bar{P}_{nm} = &\frac{\sqrt{(n-m+1)(n-m+2)(n+m-1)(n+m)}}{4}\bar{P}_{n,m-2} \\ &-\frac{(n+m)(n-m+1)+(n-m)(n+m+1)}{4}\bar{P}_{nm} \\ &-\frac{(n+m)(n-m+1)+(n-m)(n+m+1)}{4}\bar{P}_{nm} \\ &+\frac{\sqrt{(n-m-1)(n-m)(n+m+1)(n+m+2)}}{4}\bar{P}_{n,m+2}, \quad m>2\end{aligned} \tag{3.14}$$

$$\frac{\partial^2}{\partial\theta^2}\bar{P}_{00}(t) = 0, \quad \frac{\partial^2}{\partial\theta^2}\bar{P}_{10}(t) = -\sqrt{3}t, \quad \frac{\partial^2}{\partial\theta^2}\bar{P}_{11}(t) = -\sqrt{3}u \tag{3.15}$$

**Boundary value correction for ellipsoid and spherical boundary surface**

(1) Ellipsoid correction of gravity. The correction of the gravity $g$ on an ellipsoid surface outside the Earth from the vertical direction to the normal gravity direction, also known as the vertical deflection correction of gravity.

$$\varepsilon_p = \gamma sin\theta cos\theta\left[3J_2\left(\frac{a}{r}\right)^2 + \frac{\omega^3r^3}{GM}\right]\xi \tag{4.1}$$

(2) The correction of the gravity $g$ from the normal gravity direction to the geocentric direction

$$\varepsilon_h = \gamma e^2 sin\theta cos\theta\xi \tag{4.2}$$

(3) The correction of the normal gravity $\gamma$ from the normal gravity direction to the geocentric direction

$$\varepsilon_\gamma = 3\gamma\left[J_2\frac{a^2}{r^3}(3cos^2\theta-1) - \frac{\omega^3r^3}{GM}sin^2\theta\right]T \tag{4.3}$$

When the boundary surface is an ellipsoidal surface, only one ellipsoid correction is required in equation (4.1). When the boundary surface is spherical surface, it is necessary to perform three items boundary value corrections using equations (4.1) ~ (4.3).