Various terrain effects on all-element gravity field in whole Earth space

7.5 Terrain effect algorithms on various gravity field elements outside geoid

The theory of the Earth's gravitational field points out that any type of anomalous gravity field element outside the Earth can be expressed as a linear combination of the disturbing potential, gravity disturbance or their partial derivatives on some an equipotential surface. For example, the vertical deflection can be expressed by the local horizontal partial derivative of the disturbing potential, and the disturbing gravity gradient can be expressed by the vertical derivative of the gravity disturbance. Therefore, if we can get the terrain effects on disturbing potential and gravity disturbance, we also can get the terrain effects on other types of gravity field elements since there is no terrain effect in normal gravity field..

7.5.1 Expression of land terrain mass gravitational field - land complete Bouguer effect

The land terrain mass gravitational field is also called as as the complete Bouguer effect of land terrain, which is defined as the effect of the terrain mass above the geoid on the Earth's gravity field.

(1) Land complete Bouguer effect on disturbing potential

Ignoring the mass effect of the Earth's atmosphere, the disturbing potential T at the calculation point outside the Earth can be expressed as the sum of the terrain mass gravitational potential T^t (land complete Bouguer effect) and the disturbing potential T^{NT} after the terrain removed:

$$T = T^{NT} + T^t = T^{NT} + T^B + T^R \tag{5.1}$$

where T^t is the gravitational potential generated by the total terrain mass at the calculation point, which is called as the terrain effect on the disturbing potential or land complete Bouguer effect. T^R is the effect of the local terrain mass on the gravitational potential at the calculation point, called as the local terrain effect on the disturbing potential. T^B is the gravitational potential at the calculation point generated by the mass of the spherical shell with a thickness equal to the terrain height, which is called as the spherical shell Bouguer effect on the disturbing potential.

From the harmonic properties of the disturbing potential T outside the Earth, it can be known that the land complete Bouguer effect T^t, local terrain effect T^R and spherical shell Bouguer effect T^B on the disturbing potential outside the Earth are all harmonic.

Under the spherical approximation, the complete Bouguer effect on disturbing potential in the near-Earth space outside the Earth ($r \geq R + h$, R is the mean radius of the Earth) can be expressed as:

$$T^t = T^B + T^R = 4\pi G\rho_0 \frac{R^2 h}{r}\left(1 + \frac{h}{R} + \frac{h^2}{3R^2}\right) + T^R \tag{5.2}$$

where G is the gravitational constant, h is the terrain height directly below the calculation point, r is the geocentric distance of the calculation point, ρ_0 is the geometric mean density of the terrain from the ground to the geoid, and ρ_0 = 2.67×10³kg/m³.

(2) Land complete Bouguer effect on gravity disturbance

Substituting (5.1) into the definition of gravity disturbance, we get

$$\delta g = -\frac{\partial T^{NT}}{\partial r} - \frac{\partial T^{t}}{\partial r} = \delta g^{NT} + \delta g^{t} = \delta g^{NT} + \delta g^{B} + \delta g^{R} \tag{5.3}$$

where δg^t is called as the complete Bouguer effect on gravity disturbance, δg^B is the spherical shell Bouguer effect, and δg^R is called as the local terrain effect on gravity disturbance.

Under spherical approximation, the complete Bouguer effect on gravity disturbance:

$$\delta g^t = \delta g^B + \delta g^R = 4\pi G\tilde{\rho}\frac{R^2 h}{r^2}\left(1 + \frac{h}{R} + \frac{h^2}{3R^2}\right) + \delta g^R \tag{5.4}$$

Equations (5.2) and (5.4) are truncated to the quadratic term of h/R, which are suitable on ground and near-Earth space (such as aviation altitude), but not suitable on satellite altitude.

7.5.2 Integral formula of local terrain effect outside the Earth

(1) The local terrain effect on disturbing potential

According to the definition, only considering the surface density ρ, the local terrain effect on disturbing potential can be expressed as:

$$T^R = \gamma\zeta^R = G\rho\iint_S \int_{R+h}^{R+h'} L^{-1}(r,\psi,r')\,dr'\,ds \tag{5.5}$$

where $ds = r'^2 cos\varphi' d\varphi' d\lambda'$ is the integral areal element on the ground, and $L = \sqrt{r^2 + r'^2 - 2rr'cos\psi}$ is the space distance from the move point (namely the integral volume element $dV = dr'ds$) to the calculation point.

$$\int L^{-1}(r,\psi,r')\,dr' = ln(r' - rt + L) + C \tag{5.6}$$

where $t = cos\psi$, C is the integral constant.

When the calculation point is the same as the move point, the integral of local terrain effect on disturbing potential is singular:

$$T^R|_0 = \frac{1}{6}G\rho_0 A_0\sqrt{A_0/\pi}\left(h_{xx} + h_{yy}\right) \tag{5.7}$$

where ρ_0 is the terrain density at the calculation point, A_0 is the area of the integral areal element at the calculation point, and h_{xx}, h_{yy} are the second-order horizontal partial derivatives of the terrain at the calculation point in the north direction x and the east direction y.

(2) The local terrain effect on gravity disturbance

According to the definition, the local terrain effect on gravity disturbance can be expressed as:

$$\delta g^R = -T_r^R = -\frac{\partial T^R}{\partial r} = -G\rho\iint_S \int_{R+h}^{R+h'} \frac{\partial L^{-1}(r,\psi,r')}{\partial r}dr'ds \tag{5.8}$$

where, $\int \frac{\partial L^{-1}(r,\psi,r')}{\partial r}dr' = -\int \frac{r - r't}{L^3}dr' = -\frac{r'}{rL} + C$ (5.9)

When the calculation point is the same as the move point, the integral of local terrain effect on gravity disturbance is singular:

$$\delta g^R|_0 = \frac{1}{2}G\rho_0\sqrt{\pi A_0}\left(h_x^2 + h_y^2\right) \tag{5.10}$$

where $\left(h_x, h_y\right)$ is the terrain slope vector at the calculation point

(3) The local terrain effect on vertical deflection

Considering $\frac{\partial\psi}{\partial\varphi} = -cos\alpha,\quad \frac{\partial\psi}{\partial\lambda} = -cos\varphi sin\alpha$, we have:

$$\xi^R = \frac{T_\theta^R}{\gamma r} = -\frac{\partial T^R}{\gamma r\partial\varphi} = -\frac{\partial T^R}{\gamma r\partial\psi}\frac{\partial\psi}{\partial\varphi} = \frac{\partial T^R}{\gamma r\partial\psi}cos\alpha$$

$$= \frac{G\rho}{\gamma r}\iint_s \int_{R+h}^{R+h'} \frac{\partial L^{-1}(r,\psi,r')}{\partial\psi}dr'cos\alpha ds \tag{5.11}$$

$$\eta^R = -\frac{T_\lambda^R}{\gamma r sin\theta} = -\frac{\partial T^R}{\gamma r cos\varphi\partial\lambda} = -\frac{\partial T^R}{\gamma r cos\varphi\partial\psi}\frac{\partial\psi}{\partial\lambda} = \frac{\partial T^R}{\gamma r cos\varphi\partial\psi}cos\varphi sin\alpha$$

$$= \frac{G\rho}{\gamma r}\iint_s \int_{R+h}^{R+h'} \frac{\partial L^{-1}(r,\psi,r')}{\partial\psi}dr'sin\alpha ds \tag{5.12}$$

where $\int \frac{\partial L^{-1}(r,\psi,r')}{\partial\psi}dr' = \frac{r-r't}{Lsin\psi} + C,$ (5.14)

α is the geodetic azimuth of ψ, which can be obtained from the spherical trigonometric formula:

$$sin\psi cos\alpha = cos\varphi sin\varphi' - sin\varphi cos\varphi' cos(\lambda' - \lambda) \tag{5.14}$$

$$sin\psi sin\alpha = cos\varphi' sin(\lambda' - \lambda) \tag{5.15}$$

(4) The local terrain effect on disturbing gravity gradient

$$T_{rr}^R = \frac{\partial^2}{\partial r^2}T^R = G\rho\iint_s \int_{R+h}^{R+h'} \frac{\partial^2 L^{-1}(r,\psi,r')}{\partial r^2}dr'\, ds \tag{5.16}$$

where $\int \frac{\partial^2 L^{-1}(r,\psi,r')}{\partial r^2}dr' = \int\left[-\frac{1}{L^3} + \frac{3(r-r't)^2}{L^5}\right]dr' = \frac{r'}{r^2 L} + \frac{r'(r-r't)}{rL^3} + C$ (5.17)

(5) The local terrain effect on tangential gravity gradient

$$T_{NN}^R = \frac{1}{r}T_r^R + \frac{1}{r^2}T_{\theta\theta}^R = -\frac{1}{r}\delta g^R - \frac{1}{r^2}T_{\varphi\varphi}^R \tag{5.18}$$

$$T_{WW}^R = \frac{1}{r}T_r + \frac{1}{r^2}T_\theta ctg\theta + \frac{1}{r^2 sin^2\theta}T_{\lambda\lambda} = -\frac{1}{r}\delta g^R + \frac{\gamma}{r}\xi^R ctg\theta + \frac{1}{r^2 cos^2\varphi}T_{\lambda\lambda}^R \tag{5.19}$$

$$T_{\varphi\varphi}^R = \frac{\partial^2 T^R}{\partial\psi^2}\frac{\partial^2\psi}{\partial\varphi^2},\quad T_{\lambda\lambda}^R = \frac{\partial^2 T^R}{\partial\psi^2}\frac{\partial^2\psi}{\partial\lambda^2} \tag{5.20}$$

Taking the partial derivative with respect to φ on both sides of equation (5.14), considering $\frac{\partial\psi}{\partial\varphi} = -cos\alpha,\quad \frac{\partial\psi}{\partial\lambda} = -cos\varphi sin\alpha$, we have:

$$sin\psi\frac{\partial^2\psi}{\partial\varphi^2} = -\sin\varphi sin\varphi' - cos\varphi cos\varphi' cos(\lambda' - \lambda) + cos\psi cos^2\alpha \tag{5.21}$$

In the same way, taking the partial derivatives of both sides of (5.15) with respect to λ, we have: $-cos\psi cos\varphi sin^2\alpha + sin\psi\frac{\partial^2\psi}{\partial\lambda^2} = -cos\varphi' sin(\lambda' - \lambda)$, so that we can get:

$$sin\psi\frac{\partial^2\psi}{\partial\lambda^2} = -cos\varphi' sin(\lambda' - \lambda) + cos\psi cos\varphi sin^2\alpha \tag{5.22}$$

Calculate the second-order partial derivative with respect to the spherical angular distance ψ on both sides of the integral of the local terrain effect on disturbing potential, we have:

$$\frac{\partial^2 T^R}{\partial\psi^2} = G\rho\iint_s \int_{R+h}^{R+h'} \frac{\partial^2}{\partial\psi^2}\frac{1}{L}dr'ds = G\rho\iint_s \int_{R+h}^{R+h'} \frac{\partial^2}{\partial\psi^2}\frac{1}{\sqrt{r^2+r'^2-2rr'cos\psi}}dr'ds \quad (5.23)$$

$$\int \frac{\partial^2}{\partial\psi^2}\frac{1}{L}dr' = \frac{r'(6r^2+4r'^2+6r^2cos2\psi-rr'cos3\psi)-rt(4r^2+11r'^2)}{4L^3sin^2\psi} \quad (5.24)$$

7.5.3 Fast FFT algorithms of the integral of local terrain effects on various field elements

(1) Fast algorithm of the integral of local terrain effect on disturbing potential

Using the local spherical polar coordinate system, let the z-axis be the radial direction from the Earth center of mass, that is, the zenith direction, the $z = 0$ is the terrain surface, and $\tilde{h}$ is the height of the calculation point relative to the terrain surface. In this case, $dz = dr'$, $d\tilde{h} = dr$, then the local terrain effect on disturbing potential (5.5) is equivalent to:

$$T^R = G\rho\iint_s \int_0^{\Delta h}\frac{dz}{L}ds = G\rho\iint_s \int_0^{\Delta h}\frac{dz}{\sqrt{(\tilde{h}-z)^2+l^2}}ds$$

$$= G\rho\iint_s \left[ln\frac{\sqrt{(\tilde{h}-\Delta h)^2+l^2}-\tilde{h}+\Delta h}{\sqrt{(\tilde{h}-\Delta h)^2+l^2}+\tilde{h}-\Delta h} - ln\frac{\sqrt{\tilde{h}^2+l^2}-H}{\sqrt{\tilde{h}^2+l^2}+H}\right]ds \quad (5.25)$$

where Δh is the height difference of the integral move areal element ds on the surface relative to the terrain surface directly below the calculation point, $l = 2r_0 sin(\psi/2)$ is the spherical distance from the move areal element to the calculation point, and r_0 is the geocentric distance of the surface directly below the calculation point, and ψ is the spherical angular distance from the move point to the calculation point.

Expand the integrand in Eq. (5.25) to order 3 near $z = 0$, we have:

$$T^R = G\rho\iint_s \left[\frac{1}{\mathcal{L}}\Delta h + \frac{\tilde{h}}{2\mathcal{L}^3}\Delta h^2 + \frac{2\tilde{h}^2-l^2}{6\mathcal{L}^5}\Delta h^3\right]ds \quad (5.26)$$

where $\mathcal{L} = \sqrt{\tilde{h}^2 + l^2}$ is the space distance from the move areal element ds to the calculation point in the case of $z = 0$. Here $\mathcal{L} \neq L$, and L is the space distance from the move volume element $dzds$ to the calculation point.

Considering $\Delta h^2 = h'^2 - 2h'h + h^2$, $\Delta h^3 = h'^3 - 3h'^2h + 3h'h^2 - h^3$, each item on the right side of the formula (5.26) can be quickly calculated using the FFT algorithm. where, h is the terrain height directly below the calculation point, and h' is the terrain height of the move areal element.

(2) Fast algorithm of the integral of local terrain effect on gravity disturbance

In the same way, the local terrain effect on gravity disturbance (5.8) is equivalent to:

$$\delta g^R = \frac{G\rho}{r}\iint_s \left[\frac{(r_0+z)}{\sqrt{(\tilde{h}-z)^2+l^2}}\right]_0^{\Delta h} ds = \frac{G\rho}{r}\iint_s \left[\frac{r_0+\Delta h}{\sqrt{(\tilde{h}-\Delta h)^2+l^2}} - \frac{r_0}{\mathcal{L}}\right]ds \quad (5.27)$$

Expand the integrand in Eq. (5.27) to order 4 near $z = 0$, we have:

$$\delta g^R = \frac{G\rho}{r}\iint_s \left[\begin{array}{c}\frac{r\tilde{h}+\mathcal{L}^2}{\mathcal{L}^3}\Delta h + \frac{2\tilde{h}\mathcal{L}^2+r_0\left(2\tilde{h}^2-l^2\right)}{2\mathcal{L}^5}\Delta h^2 + \frac{2\tilde{h}^3 r+\tilde{h}^2 l^2-3r_0\tilde{h}l^2-l^4}{2\mathcal{L}^7}\Delta h^3 \\ +\frac{8r\tilde{h}^4-4\tilde{h}^3 l^2-12\tilde{h}l^4-24r_0\tilde{h}^2 l^2+3l^4 r_0}{8\mathcal{L}^9}\Delta h^4\end{array}\right] ds \qquad (5.28)$$

where $\Delta h^4 = h'^4 - 4h'^3 h + 6h^2 h'^2 - 4h' h^3 + h^4$。

Each item on the right side of the formula (5.28) can be quickly calculated using the FFT algorithm. In PAGravf4.5, the numerical integral of the local terrain effect on gravity disturbance is calculated using formula (5.27).

(3) Fast algorithm of the integral of local terrain effect on vertical deflection

Expand the integrand in the integral formula of the local terrain effect on vertical deflection to order 3 near $z = 0$:

$$\int_{R+h}^{R+h'} \frac{\partial L^{-1}(r,\psi,r')}{\partial\psi} dr'$$
$$= -\frac{r^2 sin\psi}{\mathcal{L}^3}\Delta h - \frac{3\tilde{h}r^2 sin\psi}{2\mathcal{L}^5}\Delta h^2 - \left[\frac{r^2 sin\psi}{3\mathcal{L}^5} + \frac{5r^2 sin\psi(2\tilde{h}^2-l^2)}{6\mathcal{L}^7}\right]\Delta h^3 \qquad (5.29)$$

Substitute (5.29) into (5.11) and (5.12) formulas, and the FFT algorithm can be employed to quickly calculate the local terrain effect on vertical deflection.

(4) Fast algorithm of the integral of local terrain effect on disturbing gravity gradient

The local terrain effect on disturbing gravity gradient (5.16) is equivalent to:

$$T_{rr}^R = G\rho \iint_s \left[\frac{\tilde{h}-\Delta h}{((\tilde{h}-\Delta h)^2+l^2)^{3/2}} - \frac{\tilde{h}}{\mathcal{L}^3}\right] ds \qquad (5.30)$$

Expand the integrand in Eq. (5.30) to order 3 near $z = 0$, we have:

$$T_{rr}^R = G\rho \iint_s \left[\begin{array}{c}\frac{2\tilde{h}^2-l^2}{\mathcal{L}^5}\Delta h - \frac{3\tilde{h}\left(2\tilde{h}^2-3l^2\right)}{2\mathcal{L}^7}\Delta h^2 + \\ \frac{4\tilde{h}^4+6r^4-12\tilde{h}^2 l^2-(6r^4+3r^2 l^2)t}{\mathcal{L}^9}\Delta h^3\end{array}\right] ds \qquad (5.31)$$

(5) Fast algorithm of the integral of local terrain effect on tangential gravity gradient

The integrand function in Equation (5.23) is equivalent to:

$$\int_{R+h}^{R+h'} \frac{\partial^2}{\partial\psi^2}\frac{1}{L}dr' = \int_0^{\Delta h}\frac{\partial^2}{\partial\psi^2}\frac{1}{\sqrt{\left(\tilde{h}-z\right)^2+4r_0^2 sin^2(\psi/2)}}dz$$
$$= \frac{1}{8sin^2\frac{\psi}{2}}\left[\frac{\tilde{h}\left(2\mathcal{L}^2+r_0^2 sin^2\psi\right)}{\mathcal{L}^3} - \frac{(\tilde{h}-\Delta h)(2\mathcal{L}^2+r_0^2 sin^2\psi-4\tilde{h}\Delta h+2\Delta h^2)}{(\mathcal{L}^2-2\tilde{h}\Delta h+\Delta h^2)^{3/2}}\right] \qquad (5.32)$$

Expand it to order 3 near z=0, we have:

$$\int_{R+h}^{R+h'} \frac{\partial^2}{\partial\psi^2}\frac{1}{L}dr' = -\frac{2(\tilde{h}^2+2r_0^2)cos\psi+r_0^2(-5+cos2\psi)}{2L^5}r_0^2\Delta h$$
$$+\frac{6(\tilde{h}^2+2r_0^2)cos\psi+3r_0^2(-7+3cos2\psi)}{4\mathcal{L}^7}\tilde{h}r_0^2\Delta h^2$$
$$+\frac{(8\tilde{h}^4+12\tilde{h}^2 r_0^2-19r_0^4)cos\psi-r_0^2(36\tilde{h}^2-18r_0^2-(24\tilde{h}^2-2r_0^2)cos2\psi+3r_0^2 cos3\psi)}{4\mathcal{L}^9}r_0^2\Delta h^3 \qquad (5.33)$$

If the calculation point is also on the terrain surface, there are $\tilde{h} = 0$， $\mathcal{L} = l$, formulas (5.25) ~ (5.33) can be greatly simplified.

7.6 Seawater Bouguer effect and land-sea residual terrain effect

7.6.1 Marine terrain gravitational field - seawater complete Bouguer effect

The marine terrain gravitational field is usually represented by the seawater complete Bouguer effect. The seawater complete Bouguer effect is defined as the effect on the Earth's gravity field (various gravity field elements) because of the density of seawater compensated to the density of land terrain.

The seawater complete Bouguer effect on disturbing potential can be directly expressed by the integral formula as:

$$T^o = G\beta \iint_s \int_{R+d}^{R} L^{-1}(r,\psi,r')\, dr'\, ds \tag{6.1}$$

where $d < 0$ is the seafloor depth, β is the seawater compensation density, equal to the difference between the terrain density and the seawater density, $\beta = \rho_0 - \rho_w$ =1.64×10³kg/m³, and L is the space distance of the move volume element of the water body ($dV' = dr'ds$) to the calculation point.

Using the local spherical polar coordinate system, let the z-axis be the radial direction from the Earth center of mass, that is, the zenith direction, $z = 0$ represents the sea level, then equation (6.1) is equivalent to:

$$T^o = G\beta \iint_s \int_d^0 \frac{dz}{L} ds = G\beta \iint_s \int_d^0 \frac{dz}{\sqrt{(\tilde{h}-z)^2+l^2}} ds$$

$$= G\beta \iint_s \left[ln\frac{\sqrt{\tilde{h}^2+l^2}-\tilde{h}}{\sqrt{\tilde{h}^2+l^2}+\tilde{h}} - ln\frac{\sqrt{(\tilde{h}-d)^2+l^2}-\tilde{h}+d}{\sqrt{(\tilde{h}-d)^2+l^2}+\tilde{h}-d} \right] ds \tag{6.2}$$

where, $ds = r'^2 d\sigma = r'^2 cos\varphi' d\varphi' d\lambda'$ is the areal element on the sea surface, $\mathcal{L} = \sqrt{\tilde{h}^2 + l^2}$ is the space distance between the move aera element ds on the sea surface and the calculation point ($\mathcal{L} \neq L$), and $\tilde{h}$ is the altitude of the calculation point. $l = 2r_0 sin\frac{\psi}{2}$ is the distance between the calculation point and the move point on the sea surface, r_0 is the mean geocentric distance of the sea surface, the mean radius R of the Earth.

In the same way, the seawater complete Bouguer effect on gravity disturbance is:

$$\delta g^o = -\frac{\partial T^o}{\partial r} = -G\beta \iint_s \int_{R+d}^{R} \frac{\partial L^{-1}(r,\psi,r')}{\partial r} dr'\, ds \tag{6.3}$$

Equation (6.3) is equivalent to

$$\delta g^o = \frac{G}{r} \iint_s \beta \int_d^0 \frac{(r_0+z)dz}{\sqrt{(\tilde{h}-z)^2+l^2}} ds = \frac{G\beta}{r} \iint_s \left[\frac{r_0}{\mathcal{L}} - \frac{r_0+d}{\sqrt{(\tilde{h}-d)^2+l^2}} \right] ds \tag{6.4}$$

Considering $\frac{\partial\psi}{\partial\varphi} = -cos\alpha$, $\frac{\partial\psi}{\partial\lambda} = -cos\varphi sin\alpha$, the seawater complete Bouguer effect on vertical deflection is equal to:

$$\xi^o = \frac{T_\theta^o}{\gamma r} = \frac{G\beta}{\gamma r} \iint_s \int_{R+d}^{R} \frac{\partial L^{-1}(r,\psi,r')}{\partial\psi} dr' cos\alpha ds \tag{6.5}$$

$$\eta^o = -\frac{T_\lambda^o}{\gamma r sin\theta} = \frac{G\beta}{\gamma r}\iint_s \int_{R+d}^{R} \frac{\partial L^{-1}(r,\psi,r')}{\partial\psi} dr' sin\alpha ds \tag{6.6}$$

The seawater complete Bouguer effect on disturbing gravity gradient is:

$$T_{rr}^o = \frac{\partial^2}{\partial r^2} T^o = G\beta \iint_s \int_{R+d}^{R} \frac{\partial^2 L^{-1}(r,\psi,r')}{\partial r^2} dr'\, ds \tag{6.7}$$

Equation (6.7) is equivalent to

$$T_{rr}^o = G\beta \iint_s \left[\frac{\tilde{h}-d}{((\tilde{h}-d)^2+l^2)^{3/2}} - \frac{h}{\mathcal{L}^3}\right] ds \tag{6.8}$$

Similarly, by expanding the integrand in the above integral formula near the sea level $z=0$, the fast FFT algorithm formula can be derived.

Expand the integrand in Eq. (6.2) to order 3 near $z=0$, we have:

$$T^o = G\beta \int_d^0 \frac{1}{L} dz ds = G\beta \iint_s \left(\frac{1}{\mathcal{L}} d + \frac{\tilde{h}}{2\mathcal{L}^3} d^2 + \frac{2\tilde{h}^2 - l^2}{6\mathcal{L}^5} d^3\right) ds \tag{6.9}$$

Expand the integrand in Eq. (6.3) to order 4 near $z=0$, we have:

$$\delta g^0 = \frac{G}{r}\iint_s \beta \left[\begin{array}{c} \frac{r\tilde{h}+\mathcal{L}^2}{\mathcal{L}^3} d + \frac{2\tilde{h}\mathcal{L}^2 + r_0\left(\tilde{h}^2+\mathcal{L}^2\right)}{2\mathcal{L}^5} d^2 + \frac{2\tilde{h}^3 r + \tilde{h}^2 l^2 - 3r_0\tilde{h}l^2 - l^4}{2\mathcal{L}^7} d^3 \\ + \frac{8r\tilde{h}^4 - 4\tilde{h}^3 l^2 - 12\tilde{h}l^4 - 24r_0\tilde{h}^2 l^2 + 3l^4 r_0}{8\mathcal{L}^9} d^4 \end{array}\right] ds \tag{6.10}$$

Expand the integrand in the integral formula of the seawater complete Bouguer effect on vertical deflection to order 3 near $z=0$:

$$\int_{R+d}^{R} \frac{\partial L^{-1}(r,\psi,r')}{\partial\psi} dr'$$
$$= -\frac{r^2 sin\psi}{\mathcal{L}^3} d - \frac{3\tilde{h}r^2 sin\psi}{2\mathcal{L}^5} d^2 - \left[\frac{r^2 sin\psi}{3\mathcal{L}^5} + \frac{5r^2 sin\psi(2\tilde{h}^2 - l^2)}{6\mathcal{L}^7}\right] d^3 \tag{6.11}$$

From equation (6.8), we can get the seawater complete Bouguer effect on disturbing gravity gradient:

$$T_{rr}^o = -\frac{\partial \delta g^o}{\partial r} = G\beta \iint_s \left[\begin{array}{c} \frac{2\tilde{h}^2 - l^2}{\mathcal{L}^5} d + \frac{3\tilde{h}\left(2\tilde{h}^2 - 3l^2\right)}{2\mathcal{L}^7} d^2 + \\ \frac{4\tilde{h}^4 + 6r^4 - 12\tilde{h}^2 l^2 - (6r^4 + 3r^2 l^2)cos\psi}{\mathcal{L}^9} d^3 \end{array}\right] ds \tag{6.12}$$

The items on the right side of equations (6.9) to (6.12) can be quickly calculated by the FFT algorithm. If the calculation point is also on the sea surface, with $h=0$, $\mathcal{L}=l$, formulas (6.2) ~ (6.12) can be greatly simplified.

The seawater complete Bouguer effects on various gravity field elements are relatively large, and a larger integral radius should be employed for the integral calculation, such as not less than 250km.

7.6.2 Integral algorithms of residual terrain effects on various field elements outside the geoid

The land-sea residual terrain effect is defined as the short-wave and ultra-short-wave components of the land-sea complete Bouguer effect. The residual terrain effects on various types of field elements can be calculated by firstly constructing the land-sea residual terrain model and then using the integral method.

The residual terrain model (RTM) can be obtained by subtracting the low-resolution land-sea terrain model from the high-resolution land-sea terrain model with the same grid specification.

The integral formula of residual terrain effect is similar in form to the integral formula of local terrain effect/seawater complete Bouguer effect, the difference lies in the adopted the density of the move element and the radial integral domain.

(1) Numerical Integral of residual terrain effects on various field elements outside the geoid

The residual terrain effects on disturbing potential can be directly expressed as:

$$T^{rtm} = G \iint_s \int_R^{R+\delta'} \beta' L^{-1}(r,\psi,r')\, dr'\, ds \tag{6.13}$$

where, δ', β' are the residual terrain height and density at the move areal element $ds = r'^2 cos\varphi' d\varphi' d\lambda'$, respectively. When ds is located in the land area, δ' is the residual terrain height δh, and β' is the terrain density ρ_0 (= 2.67×10^{3}kg/m^3), while when ds is located in the ocean area, δ' is the residual seafloor depth δd, β' is seawater compensation density $\rho_0 - \rho_w$ (where seawater density ρ_w =1.03×10^{3}kg/m^3).

It is not difficult to find that whether the areal element ds is in the land area or in the sea area, the residual terrain δ' could be positive or negative.

In the same way, the residual terrain effect on gravity disturbing is equal to:

$$\delta g^{rtm} = -\frac{\partial T^{rtm}}{\partial r} = -G \iint_s \beta' \int_R^{R+\delta'} \frac{\partial L^{-1}(r,\psi,r')}{\partial r} dr'\, ds \tag{6.14}$$

The residual terrain effect on vertical deflection is equal to:

$$\xi^{rtm} = \frac{G}{\gamma r} \iint_s \beta' \int_R^{R+\delta'} \frac{\partial L^{-1}(r,\psi,r')}{\partial \psi} dr' cos\alpha ds$$

$$\eta^{rtm} = \frac{G}{\gamma r} \iint_s \beta' \int_R^{R+\delta'} \frac{\partial L^{-1}(r,\psi,r')}{\partial \psi} dr' sin\alpha ds \tag{6.15}$$

The residual terrain effect on disturbing gravity gradient is equal to:

$$T_{rr}^{rtm} = \frac{\partial^2}{\partial r^2} T^{rtm} = G \iint_s \beta' \int_R^{R+\delta'} \frac{\partial^2 L^{-1}(r,\psi,r')}{\partial r^2} dr'\, ds \tag{6.16}$$

Similarly, using a local spherical coordinate system, let the z-axis be the radial direction (zenith direction), and $z = 0$ is the terrain surface/sea surface. Let $\mathcal{L} = \sqrt{\tilde{h}^2 + l^2}$ be the three-dimensional space distance between the move areal element and the calculation point, then formulas (6.13) ~ (6.16) can be rewritten as:

$$T^{rtm} = G \iint_s \beta' \int_0^{\delta'} \frac{dz}{\sqrt{(\tilde{h}-z)^2 + l^2}} ds$$

$$= G \iint_s \beta' \left[ln \frac{\sqrt{(\tilde{h}-\delta')^2 + l^2} - \tilde{h} + \delta'}{\mathcal{L} + \tilde{h} - \delta'} - ln \frac{\mathcal{L} - \tilde{h}}{\mathcal{L} + \tilde{h}} \right] ds \tag{6.17}$$

$$\delta g^{rtm} = \frac{G}{r} \iint_s \beta' \iint_0^{\Delta h} \frac{\partial}{\partial \tilde{h}} \frac{dz}{\sqrt{(\tilde{h}-z)^2 + l^2}} ds = \frac{G}{r} \iint_s \beta' \left[\frac{1}{\sqrt{(\tilde{h}-\Delta h)^2 + l^2}} - \frac{1}{\mathcal{L}} \right] ds \tag{6.18}$$

$$\int_R^{R+\delta'} \frac{\partial L^{-1}(r,\psi,r')}{\partial\psi} dr' = \frac{1}{2} ctg \frac{\psi}{2} \left[\frac{\tilde{h}-\delta'}{\sqrt{(\tilde{h}-\delta')^2+l^2}} - \frac{\tilde{h}}{\mathcal{L}} \right] \tag{6.19}$$

$$T_{rr}^{rtm} = G \iint_s \beta' \left[\frac{\tilde{h}-\delta'}{((\tilde{h}-\delta')^2+l^2)^{3/2}} - \frac{\tilde{h}}{\mathcal{L}^3} \right] ds \tag{6.20}$$

(2) Fast FFT algorithms of the integral of residual terrain effect on various field elements

The integrand in the above integral formula is expanded near z=0, where z=0 is the terrain surface/sea surface.

Expand the integrand in Eq. (6.17) to order 3 near $z = 0$, we have:

$$T^{rtm} = -G \iint_s \beta' \left(\frac{1}{\mathcal{L}} \delta' + \frac{\tilde{h}}{2\mathcal{L}^3} \delta'^2 + \frac{2\tilde{h}^2-l^2}{6\mathcal{L}^5} \delta'^3 \right) ds \tag{6.21}$$

Expand the integrand in Eq. (6.18) to order 4 near $z = 0$, we have:

$$\delta g^{rtm} = \frac{G}{r} \iint_s \beta' \left[\frac{\tilde{h}}{\mathcal{L}^3} \delta' + \frac{2\tilde{h}^2-l^2}{2\mathcal{L}^5} \delta'^2 + \frac{\tilde{h}(2\tilde{h}^2-3l^2)}{2\mathcal{L}^7} \delta'^3 + \frac{8\tilde{h}^4-24\tilde{h}^2 l^2+3l^4}{8\mathcal{L}^9} \delta'^4 \right] ds \tag{6.22}$$

Expand the integrand in the integral formula (6.19) of the residual terrain effect on vertical deflection to order 3 near $z = 0$:

$$\int_R^{R+\delta'} \frac{\partial L^{-1}(r,\psi,r')}{\partial\psi} dr'$$
$$= -\frac{r^2 sin\psi}{\mathcal{L}^3} \delta' - \frac{3\tilde{h} r^2 sin\psi}{2\mathcal{L}^5} \delta'^2 - \left[\frac{r^2 sin\psi}{3\mathcal{L}^5} + \frac{5r^2 sin\psi(2\tilde{h}^2-l^2)}{6\mathcal{L}^7} \right] \delta'^3 \tag{6.23}$$

Expand the integrand in Eq. (6.20) to order 4 near z=0, we have:

$$T_{rr}^{rtm} = G \iint_s \beta' \left[\frac{2\tilde{h}^2-l^2}{\mathcal{L}^5} \delta' + \frac{3\tilde{h}(2\tilde{h}^2-3l^2)}{2\mathcal{L}^7} \delta'^2 + \frac{8\tilde{h}^4-24\tilde{h}^2 l^2+3l^4}{2\mathcal{L}^9} \delta'^3 \right] ds \tag{6.24}$$

7.6.3 Spherical harmonic analysis and synthesis of land-sea terrain masses

(1) The terrain areal density $q(\varphi,\lambda)$ of any point P(R,φ,λ) on the land-sea surface can be expressed as:

$$q(\varphi,\lambda) = \beta h = R \sum_{n=1}^{\infty} \sum_{m=0}^{n} [A_{nm} cosm\lambda + B_{nm} sinm\lambda] \bar{P}_{nm}(sin\varphi) \tag{6.25}$$

where R is the mean radius of the Earth (PAGravf4.5 replaces R with the semimajor axis a of the Earth to facilitate the combination with the geopotential model), and A_{nm}, B_{nm} are the degree n order m normalized terrain mass spherical harmonic coefficients.

In formula (6.25), when P is on the land terrain surface, h is the terrain height ($h > 0$), β is the terrain density, $\beta = \rho_0$ =2.67×10³kg/m³, while when P is on the sea surface, h is the seafloor depth (h<0), β is the compensation density of seawater, equal to the difference between terrain density ρ_0 and seawater density ρ_w, $\beta = \rho_0 - \rho_w$ =1.64×10³kg/m³.

(2) The land-sea complete Bouguer effect on the gravitational potential at the point (r,θ,λ) outside geoid in spherical coordinates can be expressed by the spherical harmonic series of global land-sea terrain masses as:

$$V^{tbg}(r,\theta,\lambda) = \frac{3GM}{r\rho_e} \sum_{n=2}^{\infty} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} (A_{nm} cosm\lambda + B_{nm} sinm\lambda) \bar{P}_{nm}(cos\theta) \tag{6.26}$$

where ρ_e =5.517×10³kg/m³ is the mean density of the Earth.

(3) The land-sea residual terrain effect on the gravitational potential at the point (r,θ,λ) outside geoid in spherical coordinates can be expressed by the spherical harmonic series of global land-sea terrain masses as:

$$V^{rtm}(r,\theta,\lambda)=\frac{3GM}{r\rho_e}\sum_{n=N}^{\infty}\left(\frac{a}{r}\right)^n\sum_{m=0}^{n}(A_{nm}cosm\lambda+B_{nm}sinm\lambda)\bar{P}_{nm}(cos\theta) \quad (6.27)$$

where N is the minimum degree of the residual terrain model.

(4) The relationship between the normalized terrain masses spherical harmonic coefficient and the normalized terrain Stokes coefficient:

$$\bar{C}_{nm}^t=\frac{3}{\rho_e}\frac{1}{2n+1}A_{nm},\quad \bar{S}_{nm}^t=\frac{3}{\rho_e}\frac{1}{2n+1}B_{nm} \quad (6.28)$$

The terrain masses between the surface and the geoid, and the seawater compensation masses between the sea surface and the seafloor together generate the terrain gravitational field, that is, the complete Bouguer effect. The terrain Stokes coefficients $\bar{C}_{nm}^t$, $\bar{S}_{nm}^t$ are the normalized spherical harmonic expansion coefficients of the terrain gravitational field., that is, the complete Bouguer effects on the Stokes coefficients of global geopotential.

7.7 Local terrain compensation and terrain Helmert condensation

7.7.1 Terrain Helmert condensation effects on various gravity field elements outside geoid

The Helmert condensation of terrestrial terrain involves a concept called as terrain masses compensation, or terrain compensation for short. Terrain compensation effect on any type of gravity field element outside the geoid is defined as the amount of mass compensation for this type of gravity field element to offset the change in the Earth's gravitational field caused by the removal of terrain mass.

The Helmert condensation process of the terrain masses can be decomposed into two steps: the first step is to deduct the gravitational field generated by the terrain masses, that is, to subtract the effect of the terrain, and the second step is to compensate for the change of the gravitational field caused by the deduction of the terrain masses, that is, to add the terrain compensate effect.

For any type of gravity field element α outside the geoid, the change of the field element caused by terrain Helmert condensation is called as the terrain Helmert condensation effect on this type of field element, which can be uniformly expressed as:

$$\alpha^h=\alpha^t-\alpha^c \quad (7.1)$$

where, α^h is the terrain Helmert condensation effect on the gravity field element α, α^t is the complete Bouguer effect on α, and α^c is the terrain compensation effect on α.

The residual terrain effect is equal to the difference between the high-resolution and low-resolution complete Bouguer effect, and similarly, the terrain Helmert condensation effect is equal to the difference between the complete Bouguer effect and the terrain compensation effect.

The space outside the geoid after terrain Helmert condensation is called as the Helmert space, and the corresponding gravitational field is called as the Helmert gravitational field, which is harmonious with one difference from the actual Earth's gravitational field due to terrain Helmert condensation.

7.7.2 Algorithm formulas of terrain compensation and Helmert condensation effect

The following presents the spherical approximation algorithms for terrain compensation effects on various gravity field elements in the near-Earth harmonic space outside the geoid.

(1) The terrain compensation effect on disturbing potential

$$T^c = T^B + T^{cR} = T^B + G \iint_s \frac{\mu' - \mu}{L} ds \tag{7.2}$$

where, T^{cR} is called as the local terrain compensation effect on disturbing potential, ds is the move areal element on the unit sphere, μ is called as the terrain compensation density, and under the spherical approximation:

$$\mu = \rho_0 h \left(1 + \frac{h}{R} + \frac{h^2}{3R^2}\right) \tag{7.3}$$

where, h is the terrain height directly below the calculation point and ρ_0 is the terrain density.

The geocentric distance is replaced by the mean geocentric distance of the calculation surface and the terrain surface, respectively, then the local terrain compensation effect integral of the second term on the right side of (7.2) can be directly calculated by the FFT algorithm.

When the calculation point is the same as the move point, the integral of compensation effect on disturbing potential is singular:

$$T^{cR}|_0 = \frac{R^2}{6\bar{r}^2} G A_0 \sqrt{A_0/\pi} \left(\mu_{xx} + \mu_{yy}\right) \tag{7.4}$$

where μ_{xx}, μ_{yy} are the second-order partial derivatives of the terrain compensation density at the calculation point in the north direction x and the east direction y.

(2) The terrain compensation effect on gravity disturbance

$$\delta g^c = \delta g^B + \delta g^{cR} = \delta g^B + G \iint_s (\mu' - \mu) \frac{r - r' t}{L^3} ds \tag{7.5}$$

where, δg^{cR} is called as the local terrain compensation effect on gravity disturbance.

When the calculation point is the same as the move point, the integral of compensation effect on gravity disturbance is singular:

$$\delta g^{cR}|_0 = \frac{R^2}{12\bar{r}^3} G A_0 \sqrt{A_0/\pi} \left(\mu_{xx} + \mu_{yy}\right) \tag{7.6}$$

(3) Calculation of the terrain Helmert condensation effect

Combining formulas (7.2) and (7.5), the terrain Helmert condensation effects on various gravity field elements outside geoid under the spherical approximation can be expressed as:

$$\alpha^h = \alpha^t - \alpha^c = (\alpha^B + \alpha^R) - (\alpha^B + \alpha^{cR}) = \alpha^R - \alpha^{cR} \tag{7.7}$$

In formula (7.7), the spherical shell Bouguer effect α^B cancels each other, so the terrain Helmert condensation effect is also equal to the difference between the local terrain effect α^R and the local terrain compensation effect α^{cR}.

Substituting formulas (5.5) and (7.4), and formulas (5.7) and (7.5) into formula (7.7), respectively, the terrain Helmert condensation effects on the disturbing potential and gravity disturbance can be obtained, and then terrain Helmert condensation effects on various other types of gravity field elements can be obtained.

(4) Fast algorithm of local terrain compensation effect on gravity disturbance

Using the local spherical polar coordinate system, let the z-axis be the radial direction (the zenith direction), in this case, $dr = d\tilde{h}$, we have

$$\begin{aligned}\delta g^{cR} &= -G \iint_s (\mu' - \mu)\frac{\partial}{\partial \tilde{h}}\frac{1}{L} ds = G \iint_s (\mu' - \mu)\frac{\tilde{h}}{L^3} - \frac{\mu' - \mu}{L^3}(h' - h) ds \\ &= G \iint_s (\mu' - \mu)\frac{\tilde{h}}{L^3} ds - G \iint_s \frac{\mu' h'}{L^3}\frac{t}{L} ds \\ &\quad + G \iint_s \frac{\mu' h}{L^3} ds + G \iint_s \frac{\mu h'}{L^3} ds - G \iint_s \frac{\mu h}{L^3} ds\end{aligned} \tag{7.8}$$

Each item on the right side of the formula (7.8) can be quickly calculated using the FFT algorithm.

(5) Fast algorithm of local terrain compensation effect on vertical deflection

Considering $\frac{\partial L^{-1}(r,\psi,r')}{\partial \psi} = \frac{rr' sin\psi}{L^3}$, $\frac{\partial \psi}{\partial \varphi} = -cos\alpha$, $\frac{\partial \psi}{\partial \lambda} = -cos\varphi sin\alpha$, we have

$$\begin{aligned}\xi^{cR} &= -\frac{\partial T^{cR}}{\gamma r \partial \varphi} = -\frac{\partial T^{cR}}{\gamma r \partial \psi}\frac{\partial \psi}{\partial \varphi} = \frac{\partial T^{cR}}{\gamma r \partial \psi} cos\alpha = \frac{G}{\gamma r}\iint_s (\mu' - \mu)\frac{\partial L^{-1}(r,\psi,r')}{\partial \psi} cos\alpha ds \\ &= \frac{G}{\gamma}\int_s (\mu' - \mu)\frac{r' sin\psi}{L^3} cos\alpha ds\end{aligned} \tag{7.9}$$

$$\begin{aligned}\eta^{cR} &= -\frac{\partial T^{cR}}{\gamma r cos\varphi \partial \psi}\frac{\partial \psi}{\partial \lambda} = \frac{\partial T^{cR}}{\gamma r \partial \psi} sin\alpha = \frac{G}{\gamma r}\iint_s \frac{\partial L^{-1}(r,\psi,r')}{\partial \psi}(\mu' - \mu) sin\alpha ds \\ &= \frac{G}{\gamma}\iint_s (\mu' - \mu)\frac{r' sin\psi}{L^3} sin\alpha ds\end{aligned} \tag{7.10}$$

(6) Fast algorithm of local terrain compensation effect on disturbing gravity gradient

$$\begin{aligned}T_{rr}^{cR} &= \frac{\partial^2}{\partial r^2} T^{cR} = G \iint_s (\mu' - \mu)\frac{\partial^2}{\partial r^2}\left(\frac{1}{L}\right) ds \\ &= G \iint_s (\mu' - \mu)\left(3\frac{r - r' cos\psi}{L^5} - \frac{1}{L^3}\right) ds\end{aligned} \tag{7.11}$$

7.8 Land-sea unified classic Bouguer and equilibrium effects

7.8.1 The classical reduction method for land Bouguer gravity anomaly

In Stokes theory, the Bouguer gravity anomaly is defined on the geoid, which is equal to the gravity anomaly on the geoid minus the effect of all terrain masses outside the geoid on the gravity at the ground point. The classical algorithm for the Bouguer gravity anomaly on the geoid is:

$$\Delta g_B = \Delta g - g^R - 2\pi G\rho h \tag{8.1}$$

where Δg is the gravity anomaly on the geoid, $-g^R$ is the classic plane terrain correction as well as g^R is equal to the plane approximation of the local terrain effect in PAGravf4.5, and $-2\pi G\rho h$ is called as the layer correction as well as $2\pi G\rho h$ is equal to the plane approximation of spherical shell Bouguer effect in PAGravf4.5.

In terrestrial mountainous area, the layer correction $-2\pi G\rho h$ is much less than zero, so the Bouguer gravity anomaly is generally less than zero.

Since the gravity observed point is generally not on the geoid, it is necessary to make downward continuation of the observed gravity from the observed point to the geoid to obtain the gravity anomaly Δg on the geoid, and then according to the algorithm formula (8.1) to calculate the classical Bouguer gravity anomaly.

In the classical gravity reduction process, the observed gravity is made downward continuation to the geoid using the space correction $-0.3086h + O(h^2)$ (mGal), which only considers the normal gravity gradient. While the actual situation is that even in hilly area with the terrain altitude of several hundred meters, the contribution of disturbing gravity gradient may reach or exceed the mGal level.

Considering that the height of the observed point can be easily and accurately measured at present, and the normal gravity at the observed point can be strictly calculated, PAGravf4.5 will not continue to employ the concept of space correction.

PAGravf4.5 firstly calculates the gravity anomaly at the observed point from the observed gravity and height, in which, the normal gravity calculated using the analytical formula (see Section 2.3.1). Then, the rigorous method is employed to obtain the analytical continuation value of the gravity anomaly from the observed point to the geoid.

From an ultra-high degree geopotential model, the difference between the model gravity anomaly at the observed point and on the geoid can be directly the analytical continuation value within the altitude of 1000m. Which is equivalent to removing the model gravity anomaly at observed point firstly and then restoring model gravity anomaly on the geoid. In mountainous area, the analytical continuation value can be furtherly improved by using the radial gradient continuation of residual gravity anomaly (see Section 2.5).

From the observed gravity and height, PAGravf4.5 calculate the classic Bouguer gravity anomaly on geoid according to the general formula in following:

$$\Delta g_B = \Delta g^s - g^R - 2\pi G\rho h - \Delta g^c \tag{8.2}$$

where Δg^s is the gravity anomaly at the observed point (see section 2.3 for the calculation method), and Δg^c is the the analytical continuation value.

Since the object affected by terrain is gravity itself, it has nothing to do with the normal gravity. Therefore, the algorithm formula for the Bouguer gravity disturbance on the geoid is:

$$\delta g_B = \delta g^s - g^R - 2\pi G\rho h - \delta g^c \tag{8.3}$$

where δg^s is the gravity disturbance at the observed point (see section 2.3 for the calculation method), and δg^c is the analytical continuation value, which is almost equal to Δg^c.

In formula (8.2) or (8.3), the observed points can be on the ground or in near-Earth space outside the ground.

It should be emphasized that no matter whether the observed point is on the ground or in the near-Earth space (such as aviation altitude), the classical Bouguer gravity anomaly and classical Bouguer gravity disturbance can only be defined on the geoid, and the terrain effect can only be the effect of terrain masses on the ground gravity. g^R in the formula (8.2) or (8.3) can only be the local terrain effect on the ground gravity, even if the observed point is at the altitude of the air.

7.8.2 Calculation of land-sea unified Bouguer gravity anomaly

The existence of terrestrial terrain makes the space outside the geoid exist mass, which need to be removed, resulting in the land complete Bouguer effect. In the marine area, the density of seawater below the sea level (geoid) is less than the terrain density, and the mass loss of seawater layer need to be compensated, resulting in the seawater complete Bouguer effect.

Referring to Section 7.5.4 for the calculation method of the seawater complete Bouguer effect on gravity (gravity anomaly or gravity disturbance), the exact integral formula is:

$$g_b^w = \frac{G\beta}{r}\iint_S \left[\frac{r_0}{\mathcal{L}} - \frac{r_0+d}{\sqrt{\left(\tilde{h}-d\right)^2+l^2}}\right] ds \qquad (8.4)$$

there $d<0$ is the seafloor depth, β is the seawater compensation density (the difference between the terrain density ρ_0 and the seawater density ρ_w), and$=\rho_0-\rho_w$ =1.64×10³kg/m³, $\tilde{h}$ is the height of the calculation point relative to the sea surface, r_0 is the geocentric distance of the sea surface directly below the calculation point, ds is the move areal element on sea surface, $\mathcal{L}$ is the space distance from the move areal element to the calculation point, and l is the spherical distance between the projection point of the calculation point on the sea surface and the move areal element ds.

Since the local terrain effect g^R in (8.1) and the seawater complete Bouguer effect g_b^w in (8.4) are both integral values of a certain range of areas, the offshore ocean gravity is affected by land terrain, and the coastal land gravity is affected by seawater, so both are not zero. It is necessary the land-sea unified Bouguer effect algorithm in the coastal zone.

The height of the sea level is equal to zero, so if the integral range of the local terrain effect includes the sea area, the contribution of the sea area to the local terrain effect is equal to zero. Similarly, the seafloor depth on land is equal to zero, so if the integral range of the seawater complete Bouguer effect includes land area, the contribution of land area to the seawater Bouguer effect is also zero. It is easy to find that the local terrain effect and seawater complete Bouguer effect are completely separated and seamlessly spliced in the integral domain. Therefore, the two integral formulas can be directly added to obtain the calculation formula for the land-sea unified Bouguer gravity anomaly and Bouguer gravity disturbance:

$$\Delta g_B = \Delta g^s - g^R - 2\pi G\rho h - g_b^w - \Delta g^c \tag{8.5}$$

$$\delta g_B = \delta g^s - g^R - 2\pi G\rho h - g_b^w - \delta g^c \tag{8.6}$$

Let $g^B = g^R + 2\pi G\rho h + g_b^w$ (8.7)

In PAGravf 4.5, g^B is called as the classical Bouguer effect (see Section 3.5). It is not difficult to see that the classical Bouguer effect g^B for gravity anomaly or gravity disturbance are unified and need not be distinguished.

7.8.3 Airy-Heiskanen terrain equilibrium effect on land

The Bouguer gravity anomaly usually has a large negative value in mountainous areas, and people therefore associate the 'excess' material with irregular undulating mountains on the crust, which may be compensated by the corresponding loss material in the magma layer below.

Let the depth from the sea level (geoid) to the magma level be the compensation depth D. The Airy-Heiskanen model believes that the lower crust is a magma layer with a density of ρ_1=3.27×10³kg/m³, and a mountain floats above the magma layer with a density of the crustal density ρ_0=2.67×10³kg/m³. The part of the mountain body above the sea level is the visible terrain. The higher the mountain body is, the deeper the part that sinks into the magma (called as the mountain root), and the mountain body and the mountain root are approximately symmetrical to the magma surface. A density difference $\Delta\rho_1 = \rho_1 - \rho_0$=0.6×10³kg/m³ is formed between the mountain body and the mountain root, which is the local density deficit in the magma layer.

Suppose that the surplus material of the terrain is filled into the depleted part below it and compensated. The compensation density is exactly equal to the depletion density $\Delta\rho_1$=0.6×10³kg/m³, and the compensation density make the gravity increase. The gravity value change caused by the compensation is the terrain equilibrium effect.

Let the terrain height be h and the mountain root depth be b, it can be known from the floating static equilibrium condition

$$b\Delta\rho_1 = \rho_0 h \quad \Longrightarrow \quad b = \frac{\rho_0}{\Delta\rho_1}h = 4.45h \tag{8.8}$$

Let the z-axis be the direction of the plumb line, then the terrain equilibrium effect is equal to

$$g_I = -G\Delta\rho_1 \iint_\sigma \int_D^{D+b} \frac{z-z'}{L^3}\, dz\, d\sigma \tag{8.9}$$

7.8.4 Calculation of land-sea unified equilibrium gravity anomaly

The ocean has a layer of low-density seawater ρ_w=1.03×10³kg/m³ and a layer of oceanic crust with a density equal to ρ_0=2.67×10³kg/m³. The self-weight of the two layers of material is less than the buoyancy of the magma, so it need supplement the material to achieve static balance, which leads to the magma material upwelling to the ocean area, the formation of mountain anti-root.

The compensation $\beta = \rho_0 - \rho_w = 1.64$×10³kg/m³ for the density deficit of the seawater layer, which produces the seawater complete Bouguer effect, has been expressed by formula (8.4). After the seawater density compensated, the static equilibrium condition of the ocean mountain anti-root becomes:

$$b'\Delta\rho_1 = \beta d \quad \Longrightarrow \quad b' = \frac{\beta}{\Delta\rho_1} d = 2.73d \tag{8.10}$$

where d is the seafloor depth.

The mass loss of the land mountain root requires mass compensation, so the land equilibrium effect and the plane Bouguer effect are roughly inverse. On the contrary, the ocean mountain anti-root is excess mass that need be removed, the ocean equilibrium effect and the seawater Bouguer effect are also roughly inverse. The ocean equilibrium effect is equal to

$$g_I^o = -G\Delta\rho_1 \iint_\sigma \int_{D-b'}^{D} \frac{z-z'}{L^3} dz\, d\sigma \tag{8.11}$$

Since the terrain equilibrium effect in (8.9) and the ocean equilibrium effect in (8.11) are both integral values of a certain range of areas, the terrain equilibrium effect is not zero in the offshore ocean area, and the ocean equilibrium effect is also not zero in the coastal land area. Thence it is necessary the land-sea unified equilibrium effect algorithm in the coastal zone.

The height of the sea level is equal to zero, so if the integral range of the terrain equilibrium effect includes the sea area, the contribution of the sea area to the terrain equilibrium effect is equal to zero. Similarly, the seafloor depth on land is equal to zero, so if the integral range of the ocean equilibrium effect includes land area, the contribution of land area to the ocean equilibrium effect is also zero. It is easy to find that the terrain equilibrium effect and the ocean equilibrium effect are completely separated and seamlessly spliced in the integral domain. Therefore, the two integral formulas can be directly added to obtain the calculation formula for the land-sea unified equilibrium gravity anomaly and equilibrium gravity disturbance:

$$\Delta g_B = \Delta g^s - g^B - g_I - g_I^o - \Delta g^c \tag{8.12}$$

$$\delta g_B = \delta g^s - g^R - g_I - g_I^o - \delta g^c \tag{8.13}$$

Let $g^I = g_I + g_I^o$ (8.14)

In PAGravf 4.5, g^I is called as the classical equilibrium effect (see Section 3.5). Similarly, the classical equilibrium effect g^I for gravity anomaly or gravity disturbance is unified and need not be distinguished.

7.8.5 Sign analysis of the land and sea Bouguer / equilibrium effect

The land layer effect is to remove the effect of terrain mass outside geoid on surface gravity, and the seawater Bouguer effect is the effect on surface gravity after seawater density compensated to terrain density. Therefore, the sign of the land layer effect is inverse to that of the seawater Bouguer effect.

The land equilibrium effect is the effect of filling the depleted part of the mountain root with excess terrain material on surface gravity, and the sign of land equilibrium effect is inverse to that of the layer effect. The ocean equilibrium effect is the effect on surface gravity after the process mass of the ocean mountain anti-root removed, and the sign of ocean equilibrium effect is inverse to that of the seawater Bouguer effect.

In PAGravf4.5, the layer effect (equal to the negative layer correction) is greater than zero, so the seawater Bouguer effect and the land equilibrium effect are less than zero, and the ocean equilibrium effect is greater than zero.

If + means greater than zero, - means less than zero, we have: the layer effect (+), sea water Bouguer effect (-), land equilibrium effect (-), and ocean equilibrium effect (+).

If described by the concept of classical terrain correction, we have: the layer correction (-), sea water Bouguer correction (+), land equilibrium correction (+), and ocean equilibrium correction (-).