Global surface load spherical harmonic analysis and load effect synthesis

8.2.1 Spherical harmonic series representation for equivalent water heights of surface loads

The non-tidal load variations of atmosphere, sea level, soil water, groundwater, lakes and glaciers in the Earth′s surface layer system can be expressed by the variations of surface equivalent water height (EWH) h_w or unit point mass load $q_w = \rho_w h_w$ (also known as surface density, ρ_w is the density of water).

Surface non-tidal load variations h_w directly cause the variations of geopotential outside the Earth, which are the direct influences $\Delta V^*(r,\theta,\lambda)$ of surface load geopotential variations and can be expressed as:

$$\Delta V^*(r,\theta,\lambda) = \frac{GM}{r}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}(\Delta\bar{C}_{nm}^{*}cosm\lambda + \Delta\bar{S}_{nm}^{*}sinm\lambda)\,\bar{P}_{nm}(cos\theta) \qquad (2.1)$$

Where, r is the geocentric distance of the calculation point, $(\Delta\bar{C}_{nm}^{*},\Delta\bar{S}_{nm}^{*})$ are the normalized geopotential coefficient variations directly caused by surface non-tidal load varaitions, that is, the direct influence of geopotential (Stokes) coefficients, which can be calculated according to the gravitation potential definition:

$$\begin{Bmatrix}\Delta\bar{C}_{nm}^{*}\\ \Delta\bar{S}_{nm}^{*}\end{Bmatrix} = \frac{3\rho_w}{4\pi a\rho_e(2n+1)}\left(\frac{r}{a}\right)^n \int_S \ h_w\,\bar{P}_{nm}(cos\theta)\begin{Bmatrix}cosm\lambda\\ sinm\lambda\end{Bmatrix} sin\theta d\theta d\lambda dr \qquad (2.2)$$

Here, $\int_S \cdot\, dS$ represents the global surface integral, $dS = sin\theta d\theta d\lambda dr$, and ρ_e is the mean density of the Earth.

The equivalent water height variation h_w at the ground point $(r_0 \approx a,\theta,\lambda)$ can also be expressed as a normalized load spherical harmonic series:

$$h_w(r_0,\theta,\lambda) = r_0\sum_{n=1}^{\infty}\left(\frac{a}{r_0}\right)^n \sum_{m=0}^{n}(\Delta\bar{C}_{nm}^{w}cosm\lambda + \Delta\bar{S}_{nm}^{w}sinm\lambda)\bar{P}_{nm}(cos\theta) \qquad (2.3)$$

Here, $\Delta\bar{C}_{nm}^{w},\Delta\bar{S}_{nm}^{w}$ are the degree-n order-m normalized load spherical harmonic coefficients.

Considering that in general, the long wave is dominant in the global surface load variations, n will not be too large, and the geocentric distance of the surface load is $r_0 \approx a$, so $(a/r)^n \approx 1$, then Formula (2.3) can be simplified as:

$$h_w = a\sum_{n=1}^{\infty}\sum_{m=0}^{n}(\Delta\bar{C}_{nm}^{w}cosm\lambda + \Delta\bar{S}_{nm}^{w}sinm\lambda)\bar{P}_{nm}(cos\theta) \qquad (2.4)$$

Comparing the formulas (2.2) and (2.4), we have:

$$\begin{Bmatrix}\Delta\bar{C}_{nm}^{*}\\ \Delta\bar{S}_{nm}^{*}\end{Bmatrix} = \frac{3\rho_w}{\rho_e}\frac{1}{2n+1}\begin{Bmatrix}\Delta\bar{C}_{nm}^{w}\\ \Delta\bar{S}_{nm}^{w}\end{Bmatrix} \qquad (2.5)$$

Formula (2.5) is the relationship between the normalized spherical harmonic coefficient variations $\{\Delta\bar{C}_{nm}^{w},\Delta\bar{S}_{nm}^{w}\}$ of surface equivalent water height variations and the direct influences of surface equivalent water height variaions to the normalized geopotential coefficient $(\Delta\bar{C}^*_{nm}, \Delta\bar{S}^*_{nm})$.

8.2.2 The normalized spherical harmonic series expansion for surface load deformation field

According to the theory of Earth's load deformation, the variations h_w of ground equivalent water height also lead to the deformation of solid Earth, which further make mass adjustment in Earth and produce associated geopotential, which indirectly causes the geopotential variations called the indirect influence of surface load variations, and can be characterized by load Love numbers or load tidal factor.

The total influence of the degree-n order-m normalized spherical harmonic coefficient $\{\Delta\bar{C}^w_{nm}, \Delta\bar{S}^w_{nm}\}$ of the ground equivalent water height variations to geopotential coefficients are equal to the sum of the direct influence and indirect influence of ground equivalent water height variations. The sum is also called the load effects on geopotential coefficient variations.

$$\begin{Bmatrix}\Delta\bar{C}_{nm}\\ \Delta\bar{S}_{nm}\end{Bmatrix} = (1+k'_n)\begin{Bmatrix}\Delta\bar{C}^*_{nm}\\ \Delta\bar{S}^*_{nm}\end{Bmatrix} = \frac{3\rho_w}{\rho_e}\frac{1+k'_n}{2n+1}\begin{Bmatrix}\Delta\bar{C}^w_{nm}\\ \Delta\bar{S}^w_{nm}\end{Bmatrix} \quad (2.6)$$

Here, k'_n is degree-n load potential Love number.

From the load spherical harmonic coefficient variations $\{\Delta\bar{C}^w_{nm}, \Delta\bar{S}^w_{nm}\}$, the spherical harmonic synthesis algorithm formula of the load effect $\Delta V(r,\theta,\lambda)$ on geopotential at the calculation point (r,θ,λ) on the ground or outside the solid Earth is calculated as follows:

$$\Delta V = \frac{GM}{r}\frac{3\rho_w}{\rho_e}\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\frac{1+k'_n}{2n+1}\sum_{m=0}^{n}(\Delta\bar{C}^w_{nm}\cos m\lambda + \Delta\bar{S}^w_{nm}\sin m\lambda)\bar{P}_{nm}(\cos\theta) \quad (2.7)$$

From the Bruns formula, the spherical harmonic synthesis formula of the load effect $\Delta\zeta(r,\theta,\lambda)$ on the height anomaly can be obtained:

$$\Delta\zeta = \frac{GM}{r\gamma}\frac{3\rho_w}{\rho_e}\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\frac{1+k'_n}{2n+1}\sum_{m=0}^{n}(\Delta\bar{C}^w_{nm}\cos m\lambda + \Delta\bar{S}^w_{nm}\sin m\lambda)\bar{P}_{nm}(\cos\theta) \quad (2.8)$$

Where, γ is the normal gravity. Similarly, the spherical harmonic synthesis formula of the load effect $\Delta g^s(r_0,\theta,\lambda)$ on the ground gravity can be obtained ⦿:

$$\Delta g^s(r_0,\theta,\lambda) = \frac{GM}{r_0^2}\frac{3\rho_w}{\rho_e}\sum_{n=1}^{\infty}\frac{n+1}{2n+1}\left(1+\frac{2}{n}h'_n-\frac{n+1}{n}k'_n\right)\left(\frac{a}{r_0}\right)^n$$
$$\sum_{m=0}^{n}(\Delta\bar{C}^w_{nm}\cos m\lambda + \Delta\bar{S}^w_{nm}\sin m\lambda)\bar{P}_{nm}(\cos\theta) \quad (2.9)$$

Here, h'_n is the degree-n load radial Love number and (r_0,θ,λ) is the spherical coordinates of the ground calculation point.

The spherical harmonic synthesis formula of the load effect $\Delta g^\delta(r,\theta,\lambda)$ on gravity (disturbance) at the calculation point (r,θ,λ) on the ground or outside the solid Earth is:

$$\Delta g^\delta(r,\theta,\lambda) = \frac{GM}{r^2}\frac{3\rho_w}{\rho_e}\sum_{n=1}^{\infty}\frac{n+1}{2n+1}(1+k'_n)\left(\frac{a}{r}\right)^n$$
$$\sum_{m=0}^{n}(\Delta\bar{C}^w_{nm}\cos m\lambda + \Delta\bar{S}^w_{nm}\sin m\lambda)\bar{P}_{nm}(\cos\theta) \quad (2.10)$$

Compared with formula (2.9), formula (2.10) does not include the influence due to ground radial displacement. Therefore, formula (2.9) is only suitable for calculating the load effect on ground gravity at calculation point fixed with the solid Earth, while formula (2.10) is suitable for calculating the load effect on gravity on the ground and outside the ground (such as aviation height, satellite height or ocean space). In order to distinguish these two cases, the formula in the case that the calculation point fixed with the Earth is marked ◉ here.

The normal gravity field is the starting datum of the anomalous Earth gravity field, and the normal gravity field elements do not change with time. Therefore, there is no difference between the tidal or non-tidal load effect on gravity, gravity disturbance and gravity anomaly.

The spherical harmonic synthesis formula of the load effect on ground tilt is ◉:

South: $$\Delta\xi^s(r_0,\theta,\lambda)=\frac{GM}{r_0^2}\frac{3\rho_w}{\gamma\rho_e}\sin\theta\sum_{n=1}^{\infty}\frac{1+k_n'-h_n'}{2n+1}\left(\frac{a}{r_0}\right)^n$$
$$\sum_{m=0}^{n}(\Delta\bar{C}_{nm}^w\cos m\lambda+\Delta\bar{S}_{nm}^w\sin m\lambda)\frac{\partial}{\partial\theta}\bar{P}_{nm}(\cos\theta) \quad (2.11)$$

West: $$\Delta\eta^s(r_0,\theta,\lambda)=\frac{GM}{r_0^2\sin\theta}\frac{3\rho_w}{\gamma\rho_e}\sum_{n=1}^{\infty}\frac{1+k_n'-h_n'}{2n+1}\left(\frac{a}{r_0}\right)^n$$
$$\sum_{m=1}^{n}m(\Delta\bar{C}_{nm}^w\sin m\lambda-\Delta\bar{S}_{nm}^w\cos m\lambda)\bar{P}_{nm}(\cos\theta) \quad (2.12)$$

The spherical harmonic synthesis formula of the load effect on vertical deflection is:

South: $$\Delta\xi(r,\theta,\lambda)=\frac{GM}{r^2}\frac{3\rho_w}{\gamma\rho_e}\sin\theta\sum_{n=1}^{\infty}\frac{1+k_n'}{2n+1}\left(\frac{a}{r}\right)^n$$
$$\sum_{m=0}^{n}(\Delta\bar{C}_{nm}^w\cos m\lambda+\Delta\bar{S}_{nm}^w\sin m\lambda)\frac{\partial}{\partial\theta}\bar{P}_{nm}(\cos\theta) \quad (2.13)$$

West: $$\Delta\eta(r,\theta,\lambda)=\frac{GM}{r^2\sin\theta}\frac{3\rho_w}{\gamma\rho_e}\sum_{n=1}^{\infty}\frac{1+k_n'}{2n+1}\left(\frac{a}{r}\right)^n$$
$$\sum_{m=1}^{n}m(\Delta\bar{C}_{nm}^w\sin m\lambda-\Delta\bar{S}_{nm}^w\cos m\lambda)\bar{P}_{nm}(\cos\theta) \quad (2.14)$$

The spherical harmonic synthesis formula of the load effect on the displacement of ground site is ◉:

East: $$\Delta e(r_0,\theta,\lambda)=-\frac{GM}{r_0\gamma\sin\theta}\frac{3\rho_w}{\rho_e}\sum_{n=1}^{\infty}\frac{l_n'}{2n+1}\left(\frac{a}{r_0}\right)^n$$
$$\sum_{m=1}^{n}m(\Delta\bar{C}_{nm}^w\sin m\lambda-\Delta\bar{S}_{nm}^w\cos m\lambda)\bar{P}_{nm}(\cos\theta) \quad (2.15)$$

North: $$\Delta n(r_0,\theta,\lambda)=-\frac{GM}{r_0\gamma}\frac{3\rho_w}{\rho_e}\sin\theta\sum_{n=1}^{\infty}\frac{l_n'}{2n+1}\left(\frac{a}{r_0}\right)^n$$
$$\sum_{m=0}^{n}(\Delta\bar{C}_{nm}^w\cos m\lambda+\Delta\bar{S}_{nm}^w\sin m\lambda)\frac{\partial}{\partial\theta}\bar{P}_{nm}(\cos\theta) \quad (2.16)$$

Radial: $$\Delta r(r_0,\theta,\lambda)=\frac{GM}{r_0\gamma}\frac{3\rho_w}{\rho_e}\sum_{n=1}^{\infty}\frac{h_n'}{2n+1}\left(\frac{a}{r_0}\right)^n$$
$$\sum_{m=0}^{n}(\Delta\bar{C}_{nm}^w\cos m\lambda+\Delta\bar{S}_{nm}^w\sin m\lambda)\bar{P}_{nm}(\cos\theta) \quad (2.17)$$

The spherical harmonic synthesis formula of the load effect on gravity gradient is:

Radial: $\Delta T_{rr}(r,\theta,\lambda) = \frac{GM}{r^3}\frac{3\rho_w}{\rho_e}\sum_{n=1}^{\infty}\frac{(n+1)(n+2)}{2n+1}(1+k'_n)\left(\frac{a}{r}\right)^n$

$$\sum_{m=0}^{n}(\Delta\bar{C}_{nm}^{w}cosm\lambda + \Delta\bar{S}_{nm}sinm\lambda)\bar{P}_{nm}(cos\theta) \quad (2.18)$$

North: $\Delta T_{NN}(r,\theta,\lambda) = -\frac{GM}{r^3}\frac{3\rho_w}{\rho_e}\sum_{n=1}^{\infty}\frac{1+k'_n}{2n+1}\left(\frac{a}{r}\right)^n$

$$\sum_{m=0}^{n}(\Delta\bar{C}_{nm}^{w}cosm\lambda + \Delta\bar{S}_{nm}^{w}sinm\lambda)\frac{\partial^2}{\partial\theta^2}\bar{P}_{nm}(cos\theta) \quad (2.19)$$

West: $\Delta T_{WW}(r,\theta,\lambda) = -\frac{GM}{r^3 sin^2\theta}\frac{3\rho_w}{\rho_e}\sum_{n=1}^{\infty}\frac{1+k'_n}{2n+1}\left(\frac{a}{r}\right)^n$

$$\sum_{m=1}^{n}m^2(\Delta\bar{C}_{nm}^{w}sinm\lambda + \Delta\bar{S}_{nm}^{w}cosm\lambda)\bar{P}_{nm}(cos\theta) \quad (2.20)$$

In the formulas (2.8) ~ (2.20), the first degree term $(n = 1)$ represents the contribution of the variations of Earth's center of mass caused by the surface load deformation to the corresponding geodetic elements, which can be called as the Earth’s center of mass variation effects on geodetic variations. The variation of Earth's center of mass plays an important role in geodesy, and the first degree term in the mentioned formulas cannot be ignored.

The load Love number of unit point mass load (1kg/m²) can be calculated using the relevant parameters of spherically symmetric non-rotating elastic Earth model. The degree-n load radial, horizontal and potential Love numbers h'_n, l'_n and k'_n are shown in Tab 2.1.

Tab 2.1 The value of load Love numbers

Degree-n	h'_n	l'_n	k'_n
1	-0.2871129880	0.1045044062	0
2	-0.9945870591	0.0241125159	-0.3057703360
3	-1.0546530210	0.0708549368	-0.1962722363
4	-1.0577838950	0.0595872318	-0.1337905897
5	-1.0911859150	0.0470262750	-0.1047617976
6	-1.1492536560	0.0394081176	-0.0903495805
7	-1.2183632010	0.0349940065	-0.0820573391
8	-1.2904736610	0.0322512320	-0.0765234897
10	-1.4309817610	0.0290225900	-0.0690776844
12	-1.5609348550	0.0271636708	-0.0638847506
14	-1.6797703790	0.0259680057	-0.0598385602
16	-1.7880882500	0.0251266737	-0.0564748883
20	-1.9754659020	0.0238986214	-0.0509272630
25	-2.1615247260	0.0225448633	-0.0452625739

30	-2.3044581340	0.0211578086	-0.0405033192
35	-2.4152406280	0.0197609745	-0.0364524519
40	-2.5028874800	0.0184188171	-0.0329970228
45	-2.5741299450	0.0171690959	-0.0300450548
50	-2.6337485520	0.0160264262	-0.0275153569
60	-2.7300189390	0.0140651027	-0.0234487653
70	-2.8076818590	0.0124702089	-0.0203629907
80	-2.8746338100	0.0111640070	-0.0179658948
90	-2.9350553590	0.0100800427	-0.0160636283
100	-2.9913054190	0.0091686192	-0.0145257169
120	-3.0965116190	0.0077267323	-0.0122109806
140	-3.1965444360	0.0066448758	-0.0105711243
150	-3.2455767690	0.0062018042	-0.0099238838
160	-3.2942117980	0.0058106942	-0.0093636844
180	-3.3907532400	0.0051551676	-0.0084470364
200	-3.4867370690	0.0046324760	-0.0077337989
250	-3.7248624300	0.0037212221	-0.0065109062
300	-3.9588101480	0.0031642726	-0.0057493979
350	-4.1853482260	0.0028105951	-0.0052320414
400	-4.4014325530	0.0025772705	-0.0048534799
450	-4.6045856190	0.0024162122	-0.0045579733
500	-4.7931516890	0.0022987082	-0.0043145187
600	-5.1234075730	0.0021315364	-0.0039191204
700	-5.3914177940	0.0020034613	-0.0035936423
800	-5.6025165630	0.0018887552	-0.0033104524
1000	-5.8875374130	0.0016743075	-0.0028324828
1500	-6.1543113080	0.0012327687	-0.0020071634
2000	-6.2038470670	0.0009427101	-0.0015226332
3000	-6.2137113920	0.0006307787	-0.0010176493
4000	-6.2144649520	0.0004731032	-0.0007634795
6000	-6.2150593160	0.0003153917	-0.0005091296
8000	-6.2153555850	0.0002365398	-0.0003819009

12000	-6.2156520860	0.0001576905	-0.0002546364
18000	-6.2158498910	0.0001051258	-0.0001697735
25000	-6.2159607070	0.0000756901	-0.0001222433
30000	-6.2160082030	0.0000630749	-0.0001018717
32768	-6.2160282710	0.0000577468	-0.0000932672
∞	-6.2091440000	0.0000000000	0.0000000000

8.2.3 The normalized associated Legendre functions and thier derivative to θ

When using (2.8) ~ (2.20) spherical harmonic synthesis formulas to calculate the load effects on geodetic variations, it is necessary to calculate the normalized associated Legendre function $\bar{P}_{nm}(cos\theta)$ and their first and second derivatives to θ. Here, let $t = cos\theta, u = sin\theta$, several fast algorithms are given directly as follows.

(1) Standard forward column recursion algorithm for $\bar{P}_{nm}(t)$ $(n < 1900)$

$$\begin{cases} \bar{P}_{nm}(t) = a_{nm}t\bar{P}_{n-1,m}(t) - b_{nm}\bar{P}_{n-2,m}(t) \quad \forall n > 1, m < n \\ \bar{P}_{nn}(t) = u\sqrt{\frac{2n+1}{2n}}\bar{P}_{n-1,n-1} \quad n > 1 \end{cases} \tag{2.21}$$

$$a_{nm} = \sqrt{\frac{(2n-1)(2n+1)}{(n+m)(n-m)}}, \quad b_{nm} = \sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(2n-3)(n+m)(n-m)}}$$

$$\bar{P}_{00}(t) = 1, \quad \bar{P}_{10}(t) = \sqrt{3}t, \quad \bar{P}_{11}(t) = \sqrt{3}u \tag{2.22}$$

(2) Improved Belikov recursion algorithm for $\bar{P}_{nm}(t)$ $(n < 64800)$

When $n = 0,1$, uses formula (2.22) to calculate $\bar{P}_{nm}(t)$. When $n \geq 2$:

$$\bar{P}_{n0}(t) = a_n t\bar{P}_{n-1,0}(t) - b_n\frac{u}{2}\bar{P}_{n-1,1}(t), \quad m = 0 \tag{2.23}$$

$$\bar{P}_{nm}(t) = c_{nm}t\bar{P}_{n-1,m}(t) - d_{nm}u\bar{P}_{n-1,m+1}(t) + e_{nm}u\bar{P}_{n-1,m-1}(t), \ m > 0 \tag{2.24}$$

$$a_n = \sqrt{\frac{2n+1}{2n-1}}, \quad b_n = \sqrt{\frac{2(n-1)(2n+1)}{n(2n-1)}} \tag{2.25}$$

$$c_{nm} = \frac{1}{n}\sqrt{\frac{(n+m)(n-m)(2n+1)}{2n-1}}, \quad d_{nm} = \frac{1}{2n}\sqrt{\frac{(n-m)(n-m-1)(2n+1)}{2n-1}} \tag{2.26}$$

When m >0:

$$e_{nm} = \frac{1}{2n}\sqrt{\frac{2}{2-\delta_0^{m-1}}}\sqrt{\frac{(n+m)(n+m-1)(2n+1)}{2n-1}} \tag{2.27}$$

ETideLoad4.5 adopts mainly the improved Belikov recursion algorithm to calculate the normalized associated Legendre functions $\bar{P}_{nm}(t)$。

(3) Cross-degree recursive algorithm for $\bar{P}_{nm}(t)$ $(n < 20000)$

When $n = 0,1$, uses formula (2.22) to calculate $\bar{P}_{nm}(t)$. When $n \geq 2$:

$$\bar{P}_{nm}(t) = \alpha_{nm}\bar{P}_{n-2,m}(t) + \beta_{nm}\bar{P}_{n-2,m-2}(t) - \gamma_{nm}\bar{P}_{n,m-2}(t) \tag{2.28}$$

$$\alpha_{nm} = \sqrt{\frac{(2n+1)(n-m)(n-m-1)}{(2n-3)(n+m)(n+m-1)}}$$

$$\beta_{nm} = \sqrt{1+\delta_0^{m-2}}\sqrt{\frac{(2n+1)(n+m-2)(n+m-3)}{(2n-3)(n+m)(n+m-1)}} \tag{2.29}$$

$$\gamma_{nm} = \sqrt{1+\delta_0^{m-2}}\sqrt{\frac{(n-m+1)(n+m-3)}{(n+m)(n+m-1)}}$$

(4) Non-singular recursive algorithm for $\frac{\partial}{\partial\theta}\bar{P}_{nm}(\cos\theta)$

$$\frac{\partial}{\partial\theta}\bar{P}_{nm}(\cos\theta) = -\sin\theta\frac{\partial}{\partial t}\bar{P}_{nm}(t) \tag{2.30}$$

$$\begin{cases} \frac{\partial}{\partial\theta}\bar{P}_{n0}(t) = -\sqrt{\frac{n(n+1)}{2}}\bar{P}_{n1}(t), \quad \frac{\partial}{\partial\theta}\bar{P}_{n1}(t) = \sqrt{\frac{n(n+1)}{2}}\bar{P}_{n0} - \frac{\sqrt{(n-1)(n+2)}}{2}\bar{P}_{n2} \\ \frac{\partial}{\partial\theta}\bar{P}_{nm}(t) = \frac{\sqrt{(n+m)(n-m+1)}}{2}\bar{P}_{n,m-1}(t) - \frac{\sqrt{(n-m)(n+m+1)}}{2}\bar{P}_{n,m+1}(t), \quad m>2 \end{cases} \tag{2.31}$$

$$\frac{\partial}{\partial\theta}\bar{P}_{00}(t) = 0, \quad \frac{\partial}{\partial\theta}\bar{P}_{10}(t) = -\sqrt{3}u, \quad \frac{\partial}{\partial\theta}\bar{P}_{11}(t) = \sqrt{3}t \tag{2.32}$$

(5) Non-singular recursive algorithm for $\frac{\partial^2}{\partial\theta^2}\bar{P}_{nm}(\cos\theta)$

$$\begin{cases} \frac{\partial^2}{\partial\theta^2}\bar{P}_{n0}(t) = -\frac{n(n+1)}{2}\bar{P}_{n0}(t) + \sqrt{\frac{n(n-1)(n+1)(n+2)}{8}}\bar{P}_{n2}(t) \\ \frac{\partial^2}{\partial\theta^2}\bar{P}_{n1}(t) = -\frac{2n(n+1)+(n-1)(n+2)}{4}\bar{P}_{n1}(t) + \frac{\sqrt{(n-2)(n-1)(n+2)(n+3)}}{4}\bar{P}_{n3}(t) \end{cases} \tag{2.33}$$

$$\frac{\partial^2}{\partial\theta^2}\bar{P}_{nm}(t) = \frac{\sqrt{(n-m+1)(n-m+2)(n+m-1)(n+m)}}{4}\bar{P}_{n,m-2}(t)$$

$$-\frac{(n+m)(n-m+1)+(n-m)(n+m+1)}{4}\bar{P}_{nm}(t)$$

$$+\frac{\sqrt{(n-m-1)(n-m)(n+m+1)(n+m+2)}}{4}\bar{P}_{n,m+2}(t), \quad m>2 \tag{2.34}$$

$$\frac{\partial^2}{\partial\theta^2}\bar{P}_{00}(t) = 0, \quad \frac{\partial^2}{\partial\theta^2}\bar{P}_{10}(t) = -\sqrt{3}t, \quad \frac{\partial^2}{\partial\theta^2}\bar{P}_{11}(t) = -\sqrt{3}u \tag{2.35}$$

8.2.4 Spherical harmonic analysis of global sea level variations and synthesis of load effects

Without loss of generality, we can always decompose the global sea water mass change and transport into two effects, one is the sea surface height variation when the sea water density does not change with time, and the other is the sea water density variation when the volume and spatial distribution of sea water remain unchanged while the sea surface height remains unchanged in the case. In the first case, the sea level variation is the total sea surface height variation acted by all factors, which obviously includes the sea surface height variation caused by the sea water temperature and salt change. This part of the sea level variation contributes more than 98 % to the global sea water mass change and transportation, and can be monitored efficiently and accurately by ocean tide gauge and satellite altimetry. However, the change of seawater density no longer includes the change of sea surface height caused by temperature and salinity effect, so its contribution to global seawater quality change and transportation is generally less than 2 %, and it is difficult to accurately measure. In most geodetic cases, sea level variations can be employed to represent global seawater mass changes and transport, while the impact of seawater density changes over time is left to other higher level of geodetic techniques (such as satellite geodetic measurements combined with on-site hydrological monitoring) to solve.

(1) Spherical harmonic analysis calculation of global sea level variations

The spherical harmonic analysis of global sea level variation can be calculated by fast Fourier algorithm using Formula (2.4). Firstly, the sea level variation grid time series in spherical coordinate system are constructed by integrating various sea surface height observation data (removing the mean sea surface height grid in a certain period of time). Then, the spherical harmonic analysis is carried out on the sea level variation grid at each sampling epoch according to the formula (2.4) to generate the load spherical harmonic coefficient model time series of sea level variation. The maximum degree number of the load spherical harmonic model depends on the spatial resolution of the sea level variation grid. The sampling epoch time of the time series of the load spherical harmonic coefficient model corresponds to the time series of the sea level variation grid one by one.

In Formula (2.4), the sea level variation is directly expressed as a linear combination of harmonic functions on the spherical surface. Therefore, the cumulative residual spherical harmonic analysis method can be effectively employed to improve the approach level of the load spherical harmonic coefficient model of sea level variation.

Fig 2.1 is the calculation results of global sea level variation spherical harmonic analysis program. The program inputs 0.5°×0.5° global sea level variation spherical coordinate grid time series, where the sea level variation grid at the first epoch time is shown in the right middle figure, and the land area is set to zero. According to Formula (2.4), the cumulative approach method is employed to construct the 360-degree sea level variation load spherical harmonic coefficient model time series, where the iterative residuals are shown in the lower right figure, and the sea level variation load spherical harmonic coefficient model at the first epoch is shown in the lower left figure.

The file header of the load spherical harmonic coefficient model (lower left figure) of sea level variation includes the geocentric gravitational constant GM ($\times10^{14}m^3/s^2$), equatorial radius a (m) of the Earth, zero-degree term $a\Delta C_{00}$ (cm), relative error Θ (%). Here, Θ is the percentage of the standard deviation of the final iteration residual to the standard deviation of the input grid. The maximum degree n of the spherical harmonic coefficient is equal to the number of global sea level variation cell-grids in the latitude direction. In the example, a 0.5°×0.5° grid model is input, corresponding to the maximum degree n=360.

GM, a are also known as the scale parameters of the spherical harmonic coefficient model in which the surface harmonic functions are defined on the spherical surface whose radius is equal to the equatorial radius of the Earth.

The three first-degree spherical harmonic coefficients (ΔC_{10}, ΔC_{11}, ΔS_{11}) represent the variations of Earth's center of mass due to global sea level variations. The zero-degree term can be controlled to a small value by adjusting the time datum.

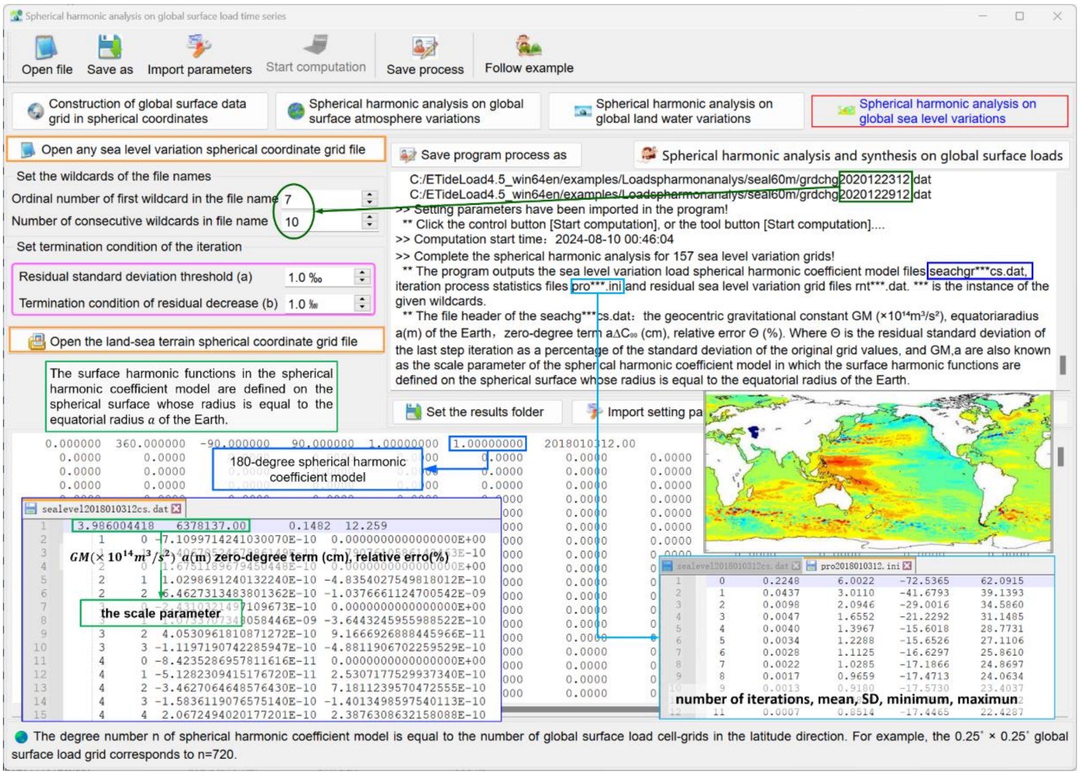

Fig 2.1 Spherical harmonic analysis on global sea level variations and construction of load spherical harmonic coefficient model

For high-precision geodesy, the contributions of the short-wave component of sea level variations cannot be ignored, and a grid model with a higher spatial resolution is needed to meet the accuracy requirements. Accordingly, a higher degree load spherical harmonic coefficient model is needed. The maximum degree number of the load spherical harmonic coefficient model can be generally determined by the global spectrum structure of the load and accuracy requirements to the load effects of sea level variations. Tab 2.2 shows the change of load spherical harmonic analysis results of global sea level variation with grid resolution (maximum degree) at a certain epoch time.

Tab 2.2 shows that the short and medium wave components of global sea level variations are obvious at this epoch time. Considering the accuracy requirements and computational efficiency, the appropriate maximum degree of the load spherical harmonic coefficient model at the epoch time can be selected as 360.

Tab 2.2 The change of spherical harmonic analysis residual of sea level variations with grid resolution

Input grid	maximum degree	Zero-degree item (cm)	First-degree item ($\times10^{-10}$)			Relative error (%)
			ΔC_{10}^{sea}	ΔC_{11}^{sea}	ΔS_{11}^{sea}	
1°×1°	180	0.1278	-7.14017	-0.74191	6.93210	6.519
30′×30′	360	0.1419	-7.29329	-0.81169	7.57094	5.075
15′×15′	720	0.1273	-7.19655	-0.71797	6.86062	3.566

(2) Spherical harmonic synthesis calculation of sea level variation load effects

From the load spherical harmonic coefficient model of sea level variations, the spherical harmonic synthesis algorithm formulas (2.8) ~ (2.20) can be employed to calculate the sea level variation load effects on all-element geodetic variations at any point on the global ground or outside the ground, and that on geopotential, gravity (acceleration) or gravity gradient outside the solid Earth such as ocean space, aviation or satellite altitude.

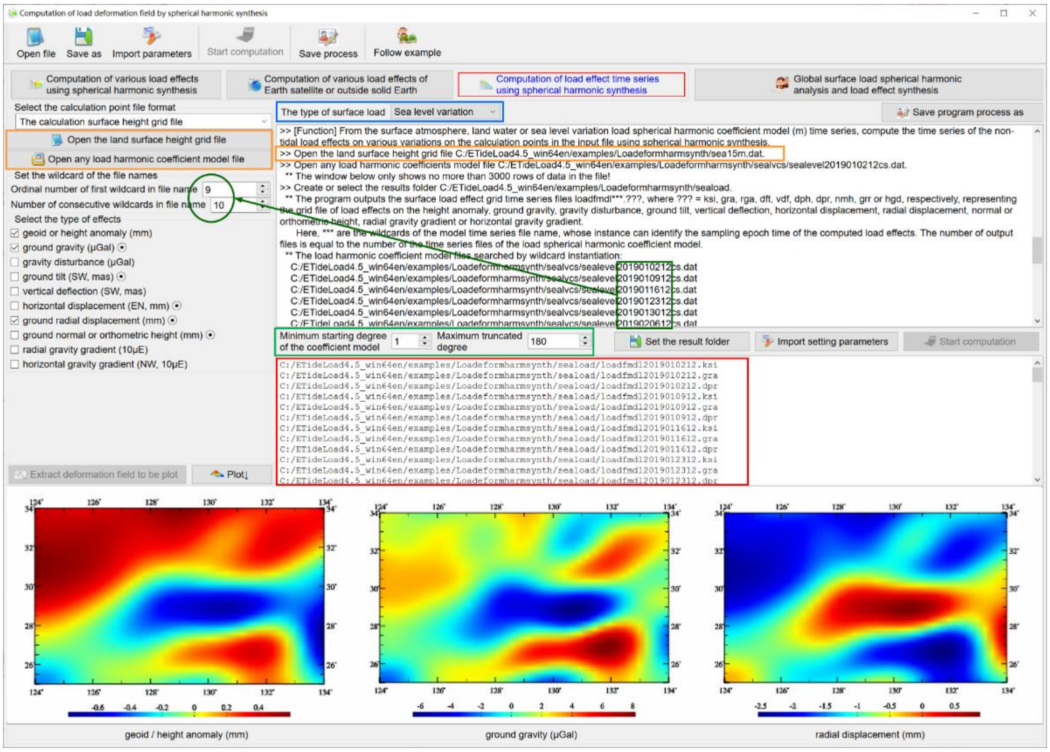

Fig 2.2 Calculation of the sea level variation load effect grid time series on all-element geodetic variations

Fig 2.2 is the calculation result of spherical harmonic synthesis program of sea level variation load effects. The program inputs the calculation area digital elevation model grid (employed to specify the area location and range of the calculation point), from the sea level variation load spherical harmonic coefficient model time series, selects the maximum calculation degree 360, and calculates the load effect grid time series on all-element geodetic variations according to formulas (2.8) ~ (2.20).

Fig 2.3 is the calculation results of the sea level variation load effects on geopotential and gravity gradient of the Earth satellite.

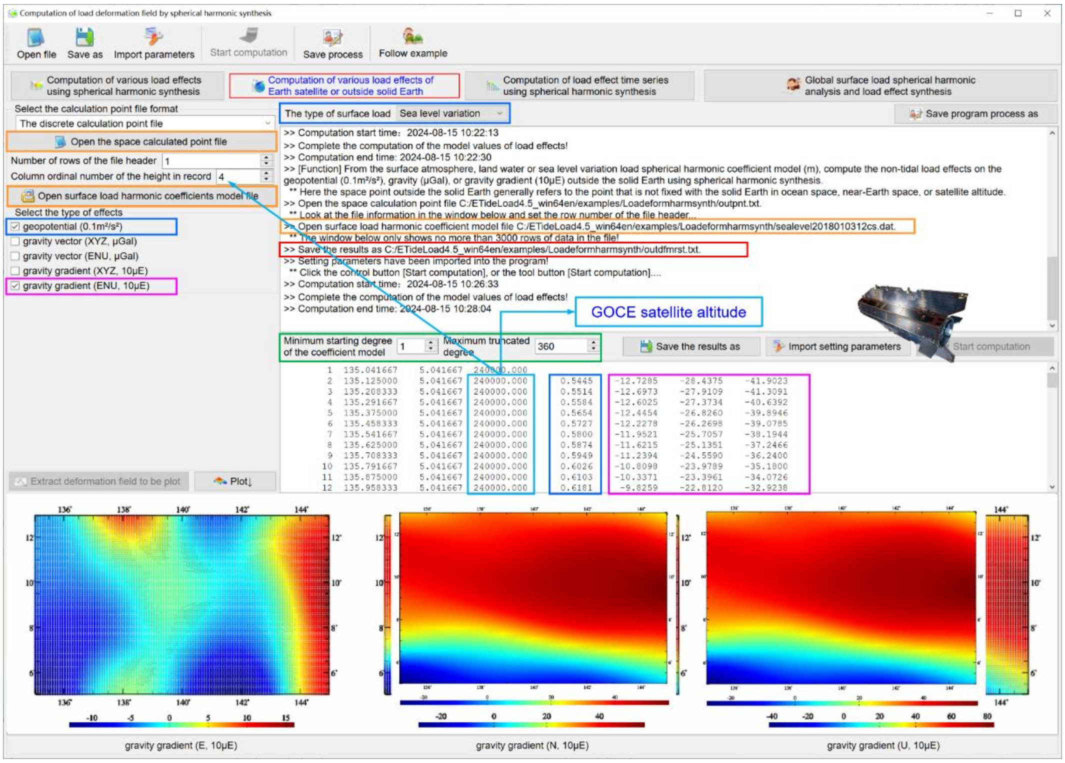

Fig 2.3 Load perturbation calculation of sea level variations of Earth satellite

In the following, the 15′×15′ global sea level weekly variations (sea level anomaly) combined by Aviso from multiple altimeter data is employed. After removing the year mean value of 2018, the 0.5°×0.5° global sea level variation (cm) spherical coordinate grid weekly time series (157 sampling epochs) from January 2018 to December 2020 are constructed. Then, the 360-degree sea level variation load spherical harmonic coefficient model (m) weekly time series are constructed by using Formula (2.4) and fast Fourier algorithm. Finally, according to the load effect spherical harmonic synthesis algorithm formulas (2.8) ~ (2.20), the sea level variation load effect weekly time series at 12 tide gauges (latitude 18°N ~ 40°N) along Chinese coast are calculated.

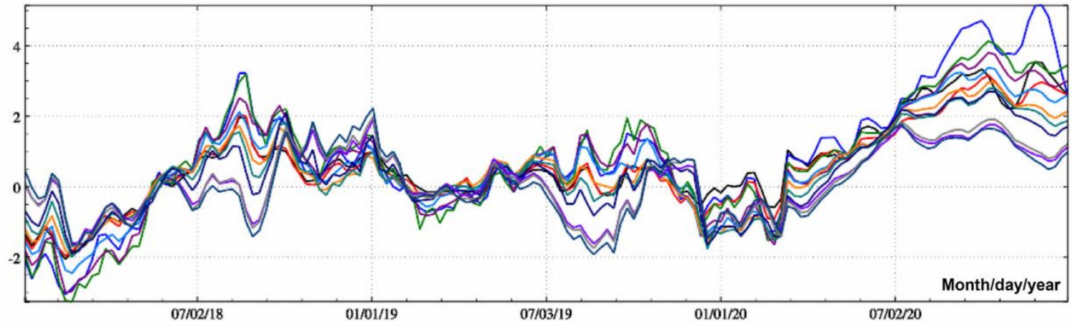

Fig 2.4 Sea level variation load effect weekly time series on the geoid (mm) at 12 tide gauges along Chinese coast

Fig 2.4 ~ Fig 2.7 are the sea level variation load effect weekly time series on the geoid (in unit of mm), ground gravity (μGal), ellipsoidal height (mm) and radial gravity gradient (10μE) respectively at 12 tide gauges from January 2018 to December 2020.

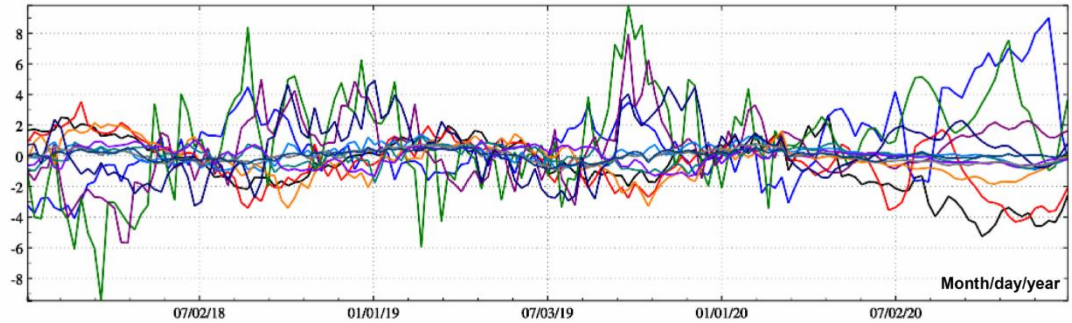

Fig 2.5 Sea level variation load effect weekly time series on the ground gravity (μGal) at 12 tide gauges along Chinese coast

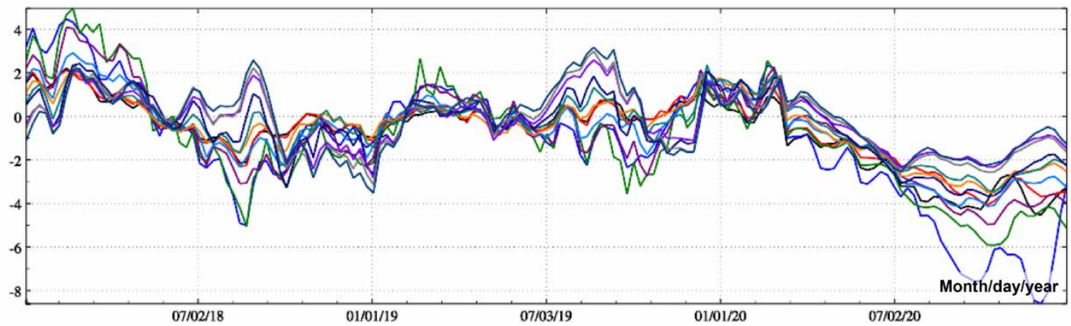

Fig 2.6 Sea level variation load effect weekly time series on the ellipsoidal height (mm) at 12 tide gauges along Chinese coast

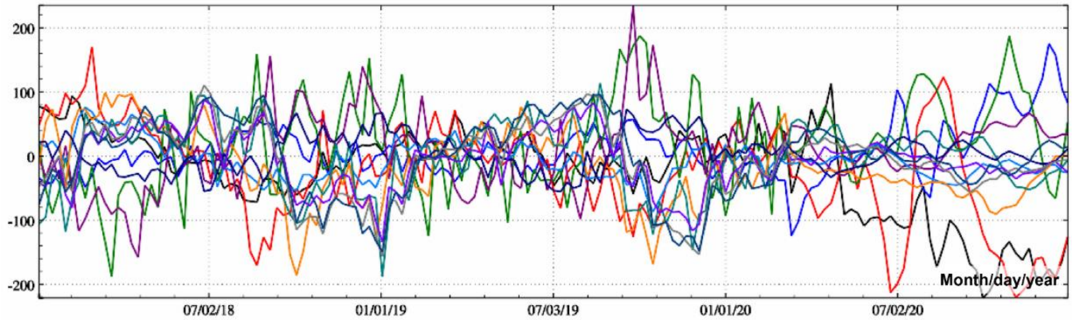

Fig 2.7 Sea level variation load effect weekly time series on the radial gravity gradient (10μE) at 12 tide gauges along Chinese coast

8.2.5 Spherical harmonic analysis of surface atmosphere variations and synthesis of load effects

(1) Load effects of atmospheric density changes and that of surface atmosphere variations

In principle, the atmospheric density load effect need three-dimensional integration of the density changes in the entire atmospheric layer space. The atmospheric load effect is usually calculated from the variation of ground atmospheric pressure by using the approximate relationship between the variation of ground atmospheric pressure and the change of atmospheric spatial density. The approximation can meet the accuracy requirements of geodesy in most cases (Guo, 2010).

A simplified calculation scheme is recommended here. When calculating the indirect influence of atmospheric load, it is assumed that the atmospheric pressure loads are concentrated on the ground, and the contribution of 1hPa (mbar) is equivalent to that of 1cm equivalent water high load, that is, 1hPa=1cm EWH, and the calculation point height h is the height of the point relative to the ground. When calculating the direct influence of atmospheric load to the gravity or radial gravity gradient, it is assumed that there is a proportional relationship between atmospheric pressure P_h at ground height h and ground atmospheric pressure P_0 as follow:

$$P_h = P_0(1 - h/44330)^{5225} \tag{2.36}$$

When calculating the load effect of atmosphere change, it is not necessary to know the atmospheric pressure P_h at the calculation point at the current epoch time. It is only necessary to determine the difference between the atmospheric pressure P_h at the current epoch and that P_h^* at the reference epoch, that is, $\Delta P_h = P_h - P_h^*$. From the difference between the ground atmospheric pressure P_0 at the current epoch and that P_0^* at the reference epoch, the atmosphere variation ΔP_h at the calculation point is obtained:

$$\Delta P_h = P_h - P_h^* = P_0\left(1 - \frac{h}{44330}\right)^{5225} - P_0^*\left(1 - \frac{h}{44330}\right)^{5225} \approx \Delta P_0\left(1 - \frac{h}{44330}\right)^{5225} \tag{2.37}$$

Using the formula (2.37), the atmospheric pressure variation ΔP_h at the height h (relative to ground) can be directly calculated from the ground atmospheric pressure change ΔP_0 without the atmospheric pressure P_0^* of the ground point at the reference epoch.

The spherical harmonic analysis process of global surface atmosphere variations is the same as that of global sea level variations. It also uses Formula (2.4) and can be calculated by fast Fourier algorithm. Firstly, global surface atmosphere variation grid time series in spherical coordinate system are constructed from the ground atmospheric pressure observations. Then, the spherical harmonic analysis is carried out for each sampling epoch of the surface atmosphere variation grid according to the formula (2.4), and the global surface atmosphere load spherical harmonic coefficient model time series are generated.

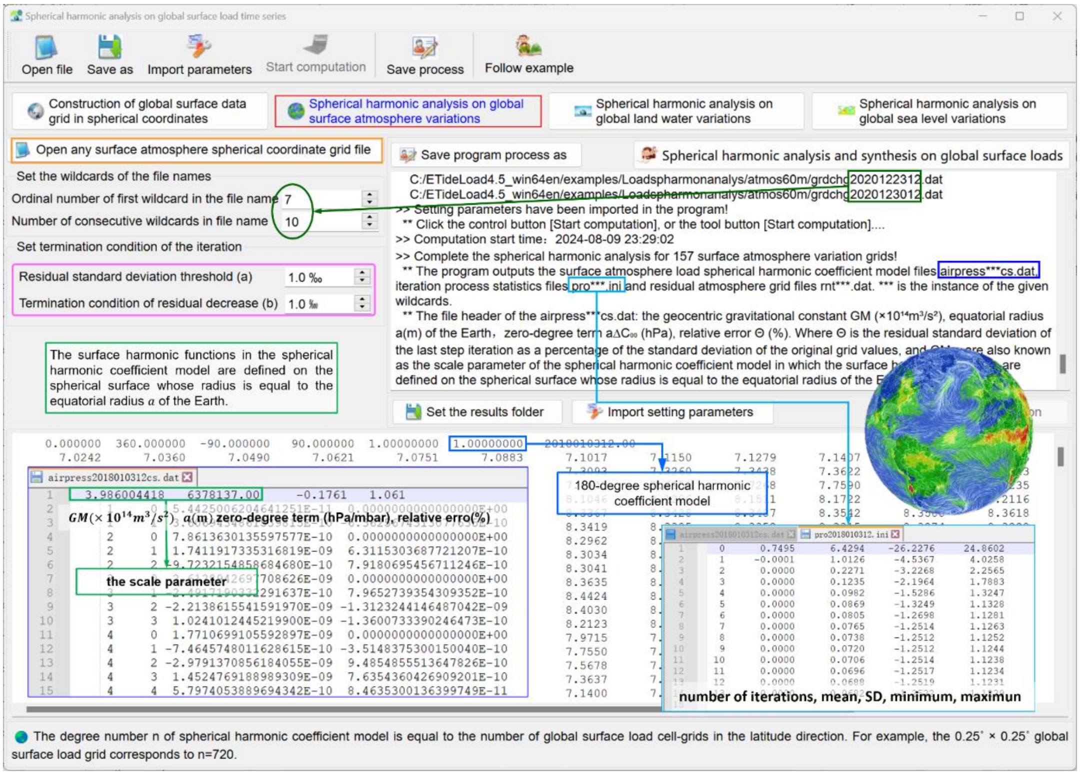

Fig 2.8 Spherical harmonic analysis on global surface atmosphere variations and construction of load spherical harmonic coefficient model

Fig 2.8 is the calculation results of surface atmosphere variation spherical harmonic analysis program. The program inputs 1°×1° surface atmosphere variation spherical coordinate grid time series. According to Formula (2.4), the cumulative approach method is employed to construct the 180-degree surface atmosphere variation load spherical harmonic coefficient model time series, where the iterative residuals are shown in the lower right figure, and the atmosphere variation load spherical harmonic coefficient model at the first epoch time is shown in the lower left figure.

Similar to the spherical harmonic analysis of sea level variations, the maximum degree number of the load spherical harmonic coefficient model can be generally determined by the global spectrum structure of the load and accuracy requirements to the load effects of surface atmosphere variations. Tab 2.3 shows the change of load spherical harmonic analysis results of global surface atmosphere variations with grid resolution (maximum degree) at a certain epoch time.

Tab 2.3 The change of spherical harmonic analysis residual of global surface atmosphere variations with grid resolution

Input grid	maximum degree	Zero-degree item (hPa)	First-degree item ($\times10^{-10}$)			Relative error (%)
			$\Delta\bar{C}_{10}^{air}$	$\Delta\bar{C}_{11}^{air}$	$\Delta\bar{S}_{11}^{air}$	
2˚×2˚	90	-1.7539	0.55043	3.60270	-6.35702	2.707
1˚×1˚	180	-1.7614	0.54424	3.60695	-8.36343	1.215
0.5˚×0.5˚	360	-1.7620	0.54251	3.60748	-8.36912	2.043

Table 3.2 shows that the long wave components are dominant in the global surface atmosphere variations at this epoch, which can be expressed by the load spherical harmonic coefficient model with the maximum degree not less than 180.

(2) Spherical harmonic synthesis calculation of surface atmosphere variation load effects

From the load spherical harmonic coefficient model of surface atmosphere variations, the spherical harmonic synthesis algorithm formulas (2.8) ~ (2.20) can be employed to calculate the surface atmosphere variation load effects on all-element geodetic variations at any point on the global ground or outside the ground, and that on geopotential, gravity (acceleration) or gravity gradient outside the solid Earth such as ocean space, aviation or satellite altitude.

Fig 2.9 is the calculation result of spherical harmonic synthesis program of surface atmosphere variation load effects. The program inputs the zero-value grid (employed to specify the area location and range of the calculation point, the zero value indicates that the calculation point is located on the ground), from the surface atmosphere variation load spherical harmonic coefficient model time series, selects the maximum calculation degree 360, and calculates the load effect grid time series on all-element geodetic variations according to formulas (2.8) ~ (2.20).

Fig 2.10 is the calculation results of the surface atmosphere variation load effects on geopotential and gravity gradient of the Earth satellite.

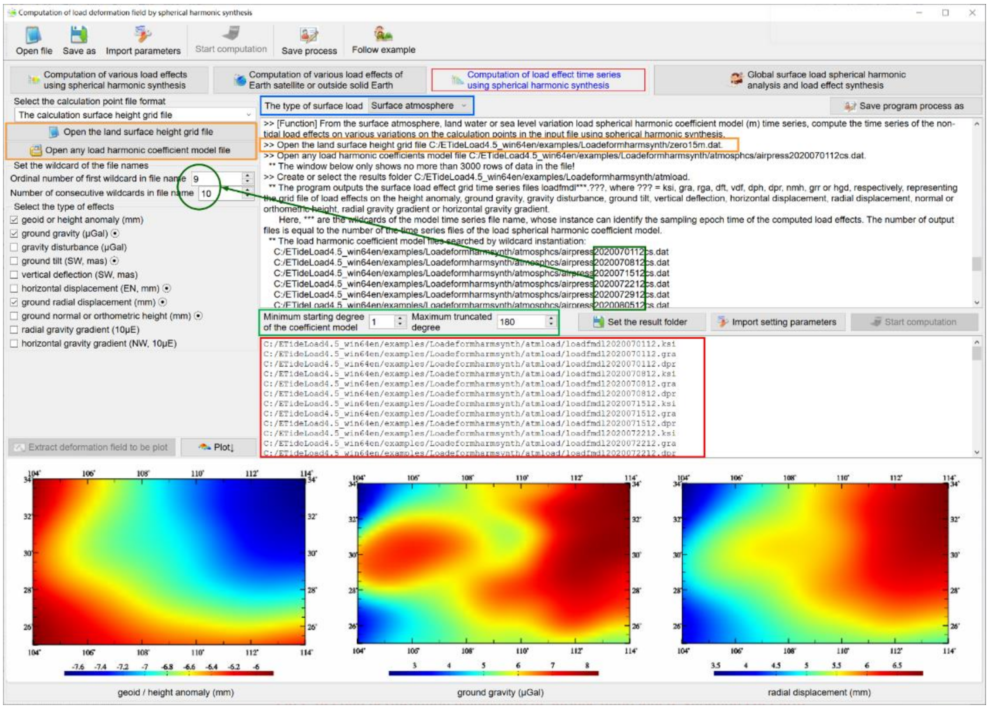

Fig 2.9 Calculation of the surface atmosphere variation load effect grid time series on all-element geodetic variations

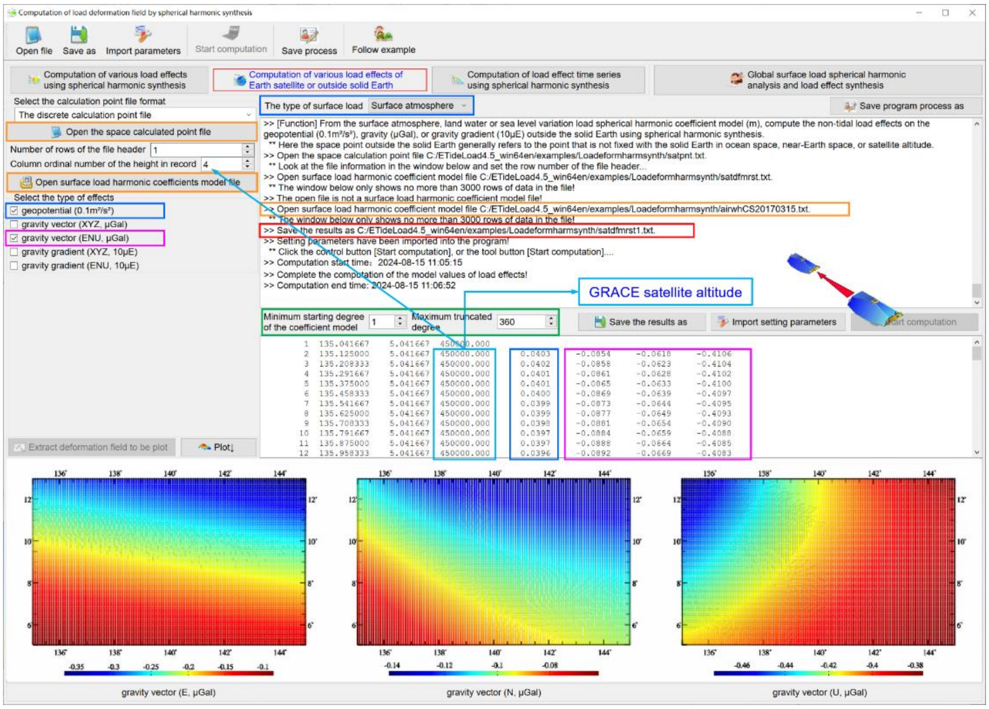

Fig 2.10 Load perturbation calculation of surface atmosphere variations of Earth satellite

In the following, the 0.5°×0.5° global surface atmospheric pressure diurnal variations in the global reanalysis data ERA-40/ERA-Interim from the European Centre for Medium-Range Weather Forecasts (ECMWF) are employed to construct the 1°×1° global surface atmosphere variation (hPa) spherical coordinate grid weekly time series (157 sampling epochs) from January 2018 to December 2020. Then, the 180-degree surface atmosphere variation load spherical harmonic coefficient model (m) weekly time series are constructed by using Formula (2.4) and fast Fourier algorithm. Finally, according to the load effect spherical harmonic synthesis algorithm formulas (2.8) ~ (2.20), the surface atmosphere load effect weekly time series at 14 CORS stations in mainland China are calculated.

Fig 2.11 ~ Fig 2.14 are the surface atmosphere variation load effect weekly time series on the geoid (in unit of mm), ground gravity (μGal), ellipsoidal height (mm) and radial gravity gradient (10μE) respectively at 14 CORS stations from January 2018 to December 2020.

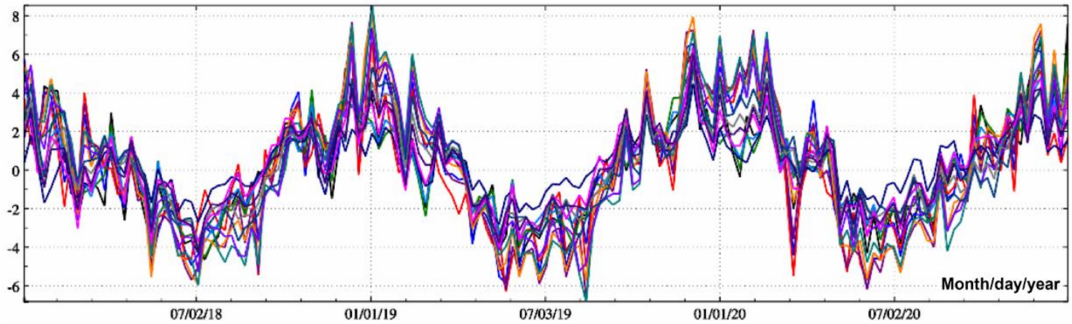

Fig 2.11 Surface atmosphere variation load effect weekly time series on the geoid (mm) at 14 CORS stations in mainland China

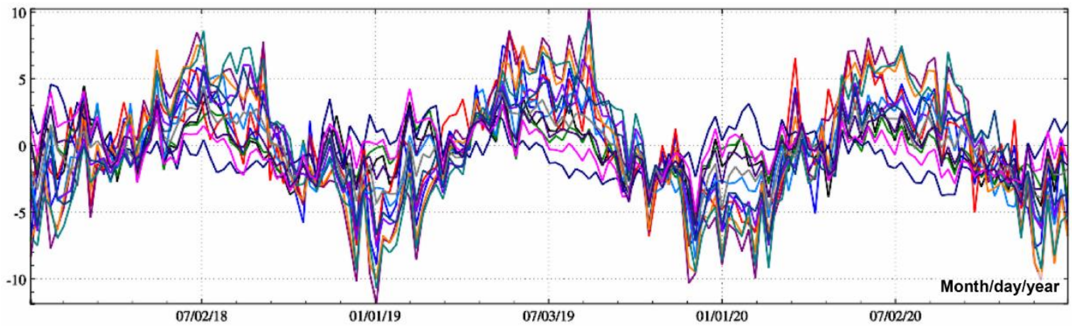

Fig 2.12 Surface atmosphere variation load effect weekly time series on the ground gravity (μGal) at 14 CORS stations in mainland China

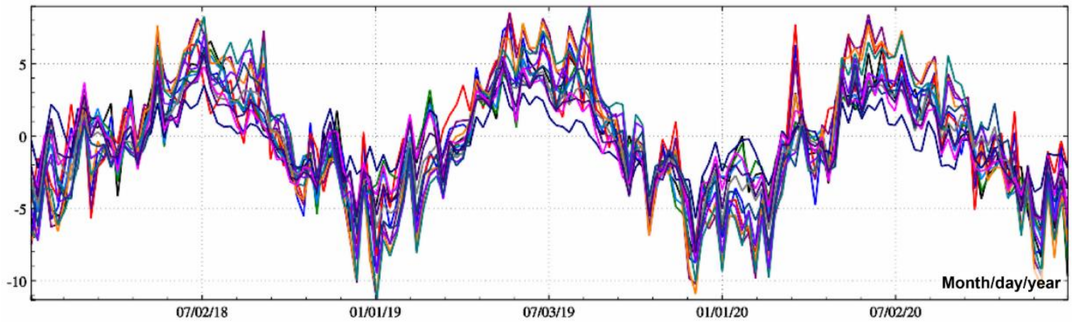

Fig 2.13 Surface atmosphere variation load effect weekly time series on the ellipsoidal height (mm) at 14 CORS stations in mainland China

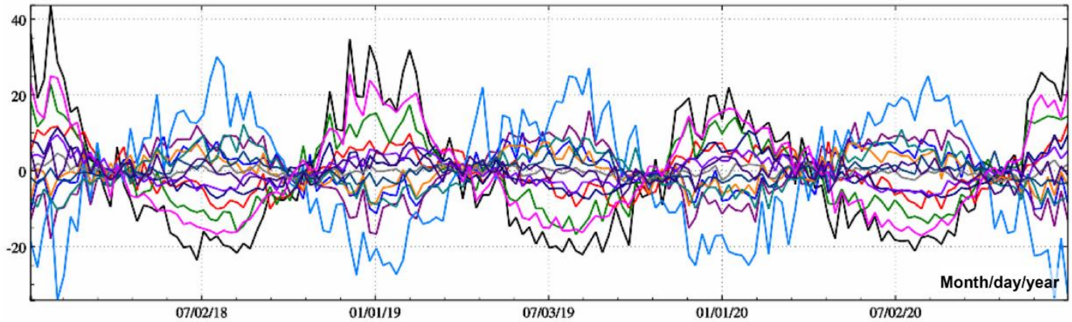

Fig 2.14 Surface atmosphere variation load effect weekly time series on the radial gravity gradient (10μE) at 14 CORS stations in mainland China

8.2.6 Spherical harmonic analysis of global land water variations and synthesis of load effects

The spherical harmonic analysis process of global land water variations expressed by the ground equivalent water height (EWH) variations is the same as that of global sea level variations. Fig 2.8 is the calculation results of land water variation spherical harmonic analysis program. The program inputs 0.25°×0.25° land water variation spherical coordinate grid (EWH equal to zero in the ocean area) time series to construct the 720-degree land water variation load spherical harmonic coefficient model time series.

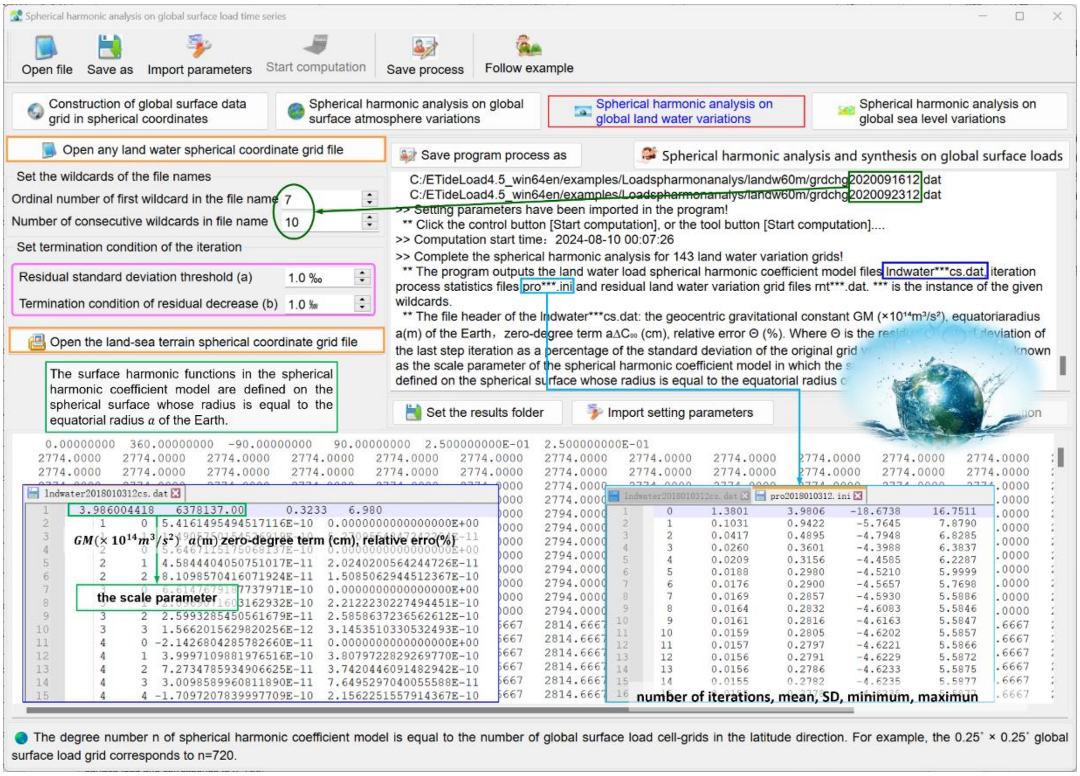

Fig 2.15 Spherical harmonic analysis on global land water variations and construction of load spherical harmonic coefficient model

Tab 2.4 shows the change of load spherical harmonic analysis results of land water variations with grid resolution (maximum degree) at a certain epoch time.

Tab 2.4 The change of spherical harmonic analysis residual of global land water variations with grid resolution

Input grid	maximum degree	Zero-degree item (cm)	First-degree item ($\times10^{-10}$)			Relative error (%)
			$\Delta\bar{C}_{10}^{lnd}$	$\Delta\bar{C}_{11}^{lnd}$	$\Delta\bar{S}_{11}^{lnd}$	
30′×30′	360	0.3242	5.46047	1.49947	0.52091	5.851
15′×15′	720	0.3207	5.32556	1.51216	0.50261	4.291
9′×9′	1200	0.3236	5.43533	1.50154	0.51493	3.094

Tab 2.4 shows that the short-wave components of global land water variations at this epoch are obvious, and the appropriate maximum degree of the load spherical harmonic coefficient model can be selected as 720.

In the following, the 0.25°×0.25° GLDAS data from NASA Global Land Data Assimilation System are employed to construct the 15′×15′ global land water variation (cm) spherical coordinate grid weekly time series (157 sampling epochs) from January 2018 to September

2020. Then, the 720-degree land water variation load spherical harmonic coefficient model (m) weekly time series are constructed by using Formula (2.4) and fast Fourier algorithm. Finally, according to the load effect spherical harmonic synthesis algorithm formulas (2.8) ~ (2.20), the land water variation load effect weekly time series at 14 CORS stations in mainland China are calculated.

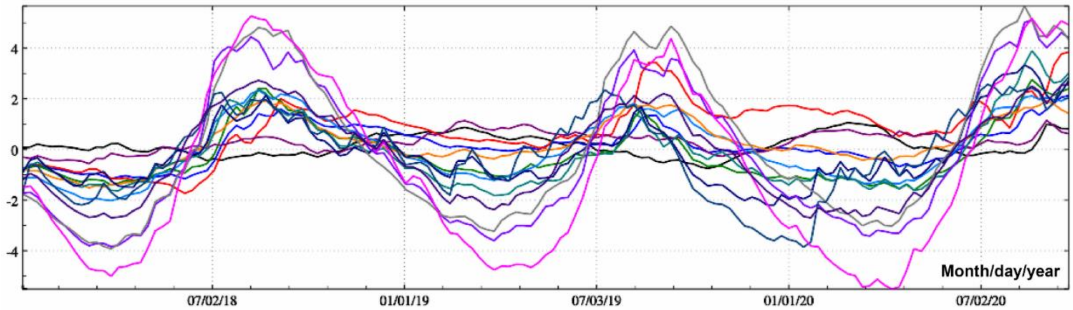

Fig 2.16 Global land water variation load effect weekly time series on the geoid (mm) at 14 CORS stations in mainland China

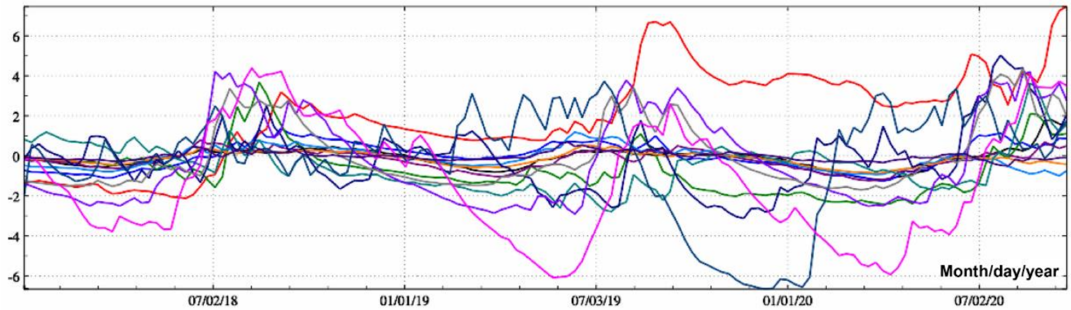

Fig 2.17 Global land water variation load effect weekly time series on the ground gravity (μGal) at 14 CORS stations in mainland China

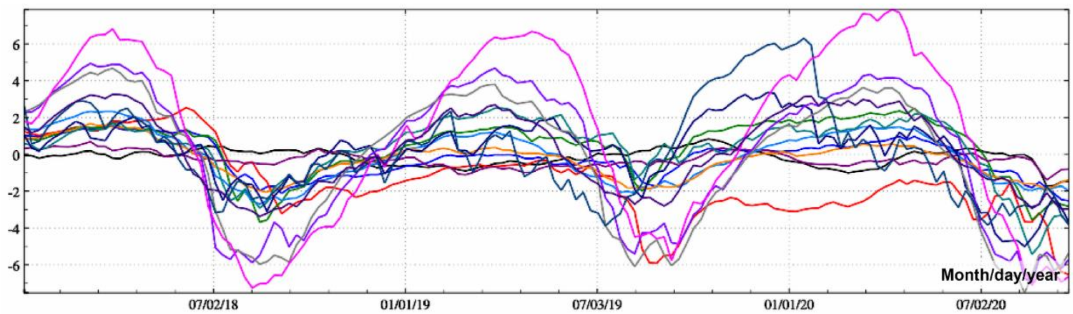

Fig 2.18 Global land water variation load effect weekly time series on the ellipsoidal height (mm) at 14 CORS stations in mainland China

Here, the land water includes 4m shallow soil water, wetland water, vegetation water and glacier snow mountain water, but does not include lake, river water and groundwater. Fig 2.16 ~ Fig 2.19 are the land water variation load effect weekly time series on the geoid (in unit of mm), ground gravity (μGal), ellipsoidal height (mm) and radial gravity gradient (10μE) respectively at 14 CORS stations from January 2018 to December 2020.

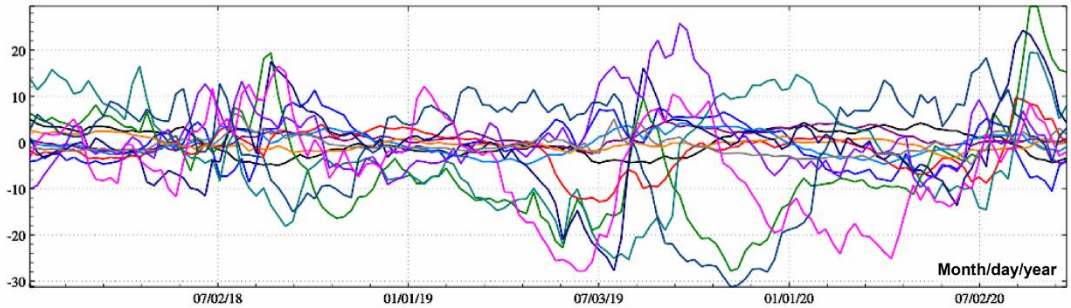

Fig 2.19 Global land water variation load effect weekly time series on the radial gravity gradient (10μE) at 14 CORS stations in mainland China

Using the monitoring data to global surface atmosphere, land water and sea level variations, determine the non-tidal temporal Earth's gravity field, as well as the non-tidal load effects on the geopotential coefficients, and then you can calibrate various parameters of the satellite's key geodetic payloads, and then effectively improve and check the quality, reliability and accuracy of the time-varying monitoring for satellite gravity field.